Serie 9

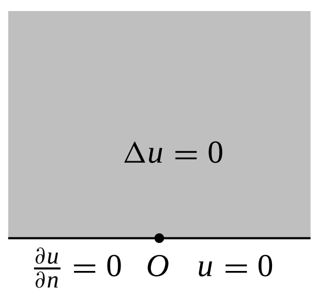
Theoretical Exercises: 1.–4.

Practical Exercise: none.

1. (singular functions)

Using the separation of variables, find the singular functions associated to the following problem:

$$\begin{cases} \Delta u = 0, & \text{for } x \in \mathbb{R} \text{ and } y > 0, \\ \dfrac{\partial u}{\partial n}(x,0) = 0, & \text{for } x < 0, \\ u(x,0) = 0, & \text{for } x > 0. \end{cases} \tag{1}$$



2. (quasi interpolation)

Let Ω be a polygonal domain of $\mathbb{R}^2$. Let $\mathcal{M} = \{K_i\}_{i=1}^M$ be a triangular mesh of Ω. Let $\{a_N^i\}_{i=1}^P$ be the nodes of $\mathcal{M}$. To each nodes a_n^i, we associate an adjacent edge $e_N^i = [a_N^i, a_N^{\sigma(i)}]$. Moreover, we suppose that this association satisfies the following rule:

$$a_N^i \in \partial\Omega \Longrightarrow e_N^i \in \partial\Omega \tag{2}$$

These vertices are parameterized with φ_N^i:

$$\begin{cases} \varphi_N^i : & [0;1] \longrightarrow e_N^i \\ & t \longmapsto a_N^i + t(a_N^{\sigma(i)} - a_N^i). \end{cases} \tag{3}$$

We define the operator T

$$\left\{\begin{array}{rcl} T: \quad H^1(\Omega) & \longrightarrow & \mathcal{S}_0^1(\mathcal{M}) \\ u & \longmapsto & \displaystyle\sum_{i=1}^{P} \int_0^1 (4-6t)\cdot u(\varphi_N^i(t))dt\ b_N^i. \end{array}\right. \tag{4}$$

where the b_N^i's, $i = 1, \ldots, P$ are the standard hat functions associated to the nodes a_N^i's.

(i) Prove that T is continuous on $H^1(\Omega)$. Is T bounded on $L^2(\Omega)$?

(ii) Prove that T is a projector i. e.

$$T^2 = T. \tag{5}$$

(iii) Show the following inclusion

$$T\left(H_0^1(\Omega)\right) \subset H_0^1(\Omega). \tag{6}$$

3. ("inverse" inequality)

Let Ω be a polygonal domain of $\mathbb{R}^2$. On a sequence of shape-regular triangular meshes, prove the following inequality:

$$\|u_N\|_{L^2(\partial\Omega)} \leqslant C\, h^{\frac{1}{2}-\frac{2}{p}}\, \|u_N\|_{L^p(\Omega)}, \quad \forall u_N \in \mathcal{S}_1^0(\mathcal{M}),\ \forall 1 \leqslant p \leqslant 2. \tag{7}$$

4. (Consistency error)

Let Ω be a polygonal of $\mathbb{R}^2$. Let $\mathcal{M}$ be a triangular mesh of Ω.

We consider the bilinear form $\mathsf{a} : \mathcal{S}_1^0(\mathcal{M}) \times \mathcal{S}_1^0(\mathcal{M}) \to \mathbb{R}$

$$\mathsf{a}(u_N, v_N) = \int_\Omega \nabla u_N\ \nabla v_N + u_N\ v_N = \sum_{K\in\mathcal{M}} \int_K \nabla u_N\ \nabla v_N + u_N\ v_N \tag{8}$$

The bilinear form a is approximated with $\mathsf{a}_N : \mathcal{S}_1^0(\mathcal{M}) \times \mathcal{S}_1^0(\mathcal{M}) \to \mathbb{R}$:

$$\mathsf{a}_N(u_N, v_N) = \sum_{K\in\mathcal{M}} \int_{\substack{K\\ num}} \nabla u_N\ \nabla v_N + u_N\ v_N \tag{9}$$

with

$$\int_{\substack{K\\ num}} f = \frac{|K|}{3} \sum_{i=1}^{3} f(a_K^i) \tag{10}$$

where the a_K^i's are the three vertices of the triangle K.

Determine the $h_{\mathcal{M}}$-asymptotic of consistency error on a sequence of shape-regular meshes.

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.