Serie 8

Theoretical Exercises: 1.–6.

Practical Exercise: none.

1. Let $\widehat{K}$ be an open set of $\mathbb{R}^d$ and let ϕ_h be the map

$$\mathbf{x} \;=\; \phi_h(\widehat{\mathbf{x}}) \;=\; h\,\widehat{\mathbf{x}}.$$

For $p \geqslant 1$ and $m \geqslant 0$, show the following scaling estimate:

$$|u|_{W^{m,p}(\phi_h(\widehat{K}))} \;=\; h^{\frac{d-mp}{p}}\, |(\phi_h)^* u|_{W^{m,p}(\widehat{K})}, \quad \forall u \in W^{m,p}(\phi_h(\widehat{K}))$$

with

$$\begin{cases} D^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \\ u \in W^{m,p}(K) \iff D^{\boldsymbol{\alpha}} u \in L^p(K), \quad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^d \text{ with } |\boldsymbol{\alpha}| \leqslant m, \\ |u|^p_{W^{m,p}(K)} \;=\; \displaystyle\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^d \;/\; |\boldsymbol{\alpha}| = m} \int_K \left(\frac{\partial^m u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \right)^m dx_1 \cdots dx_d, \\ \|u\|^p_{W^{m,p}(K)} \;=\; \displaystyle\sum_{m'=0}^{m} \|u\|^p_{W^{m',p}(K)}. \end{cases}$$

2. Let $\mathcal{M}$ be a mesh of a polygon Ω of $\mathbb{R}^2$. For $m \in \mathbb{N}_0$, we denote by $H^m(\mathcal{M})$ the Hilbert space

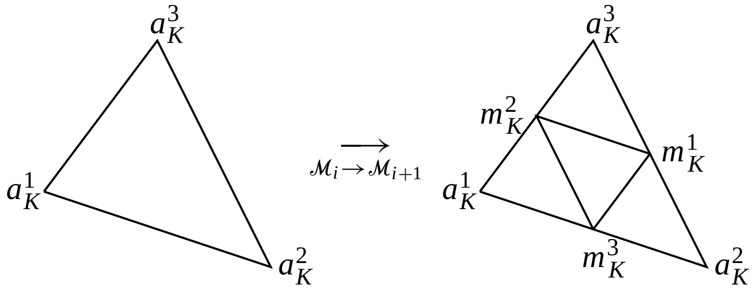
$$H^m(\mathcal{M}) = \{u \in L^2(\Omega) \;/\; u \in H^m(K), \quad \forall K \in \mathcal{M}\}$$

with scalar product:

$$(u, v)_{H^m(\mathcal{M})} \;=\; \sum_{K \in \mathcal{M}} (u, v)_{H^m(K)}.$$

For $m > k$, show the compact embedding of $H^m(\mathcal{M}) \overset{c}{\hookrightarrow} H^k(\Omega)$.

3. Let $\{\mathcal{M}_i\}_{i=0}^n$ be a sequence of triangular meshes created by regular refinement.



Let P_i —respectively Q_i— denote the $H^1(\Omega)$ —resp. $L^2(\Omega)$— projection on $\mathcal{S}_p^0(\mathcal{M}_i)$, with $p \in \mathbb{N}_0$ fixed.

Show that

$$\begin{cases} \mathsf{P}_i \underset{i \to +\infty}{\longrightarrow} \mathsf{Id} & \text{strongly in } H^1(\Omega), \\ \mathsf{Q}_i \underset{i \to +\infty}{\longrightarrow} \mathsf{Id} & \text{strongly in } L^2(\Omega), \end{cases}$$

i. e.

$$\begin{cases} \|u - \mathsf{P}_i u\|_{H^1(\Omega)} \underset{i \to +\infty}{\longrightarrow} 0, & \forall u \in H^1(\Omega), \\ \|u - \mathsf{Q}_i u\|_{L^2(\Omega)} \underset{i \to +\infty}{\longrightarrow} 0, & \forall u \in L^2(\Omega). \end{cases}$$

4. Prove the following lemma:

Lemma *Let $\mathcal{M}$ be a mesh (obtained by regular refinement) of $\Omega \subset \mathbb{R}^d$, We do equip $\mathcal{S}_1^0(\mathcal{M})$ with nodal basis b^i, $i = 1, \ldots, N := \dim \mathcal{S}_1^0(\mathcal{M})$. Then there are constants $\underline{C}, \overline{C} > 0$, only dependant on the shape functions $\widehat{b}^1, \cdots, \widehat{b}^3$ such that*

$$\underline{C} \left\| \sum_{i=1}^N \alpha_i b^i \right\|_{L^2(\Omega)}^2 \leq \sum_{i=1}^N |\alpha_i|^2 \left\| b^i \right\|_{L^2(\Omega)}^2 \leq \overline{C} \left\| \sum_{i=1}^N \alpha_i b^i \right\|_{L^2(\Omega)}^2 \qquad \forall \vec{\boldsymbol{\alpha}} \in \mathbb{C}^N.$$

5. Let Q_N be the $L^2(\Omega)$-orthogonal projection on $\mathcal{S}_p^0$ Show that

$$\|u - \mathsf{Q}_N u\|_{L^2(\Omega)} \leqslant C(m)\, h^{\min(p+1,m)}\, \|u\|_{H^m(\Omega)}, \quad \forall m \in \mathbb{N}_0.$$

implies

$$\|u - \mathsf{Q}_N u\|_{L^2(\Omega)} \leqslant C(s)\, h^{\min(p+1,s)}\, \|u\|_{H^s(\Omega)}, \quad \forall s \in \mathbb{R}_+.$$

6. We denote by Λ the interval $]-1;1[$. Let $\mathsf{Q}_p : L^2(\Lambda) \to \mathcal{P}_p(\Lambda)$ be the $L^2(\Omega)$ orthogonal projection on $\mathcal{P}_p(\Lambda)$ and let $\mathsf{I}_p : H^1(\Lambda) \to \mathcal{P}_p(\Lambda)$ be the following interpolation operator

$$\mathsf{I}_p u(x) = u(-1) + \int_{-1}^{x} (\mathsf{Q}_{p-1} u')(\xi)\, d\xi, \quad -1 \leqslant x \leqslant 1$$

On $\widehat{K} = \Lambda^2$, we denote the directional interpolation operator by I_p^x, I_p^y:

$$\begin{cases} \mathsf{I}_p^x u(x,y) \;=\; u(-1,y) \;+\; \displaystyle\int_{-1}^{x} (\mathsf{Q}_{p-1} \frac{\partial u}{\partial x}(\xi, y)) d\xi \;=\; (\mathsf{I}_p u(\cdot, y))(x), \\ \mathsf{I}_p^y u(x,y) \;=\; u(x,-1) \;+\; \displaystyle\int_{-1}^{y} (\mathsf{Q}_{p-1} \frac{\partial u}{\partial y}(x, \xi)) d\xi \;=\; (\mathsf{I}_p u(x, \cdot))(y). \end{cases}$$

Finally, we denote by $\mathbb{I}_p : H^1(\widehat{K}) \longrightarrow \mathcal{Q}_p(\widehat{K})$ the 2D interpolation operator:

$$\mathbb{I}_p \;=\; \mathsf{I}_p^x \, \mathsf{I}_p^y.$$

Prove the following error estimate:

$$|u - \mathbb{I}_p u|_{H^1(\widehat{K})} \;\leqslant\; C \; p^{-(k-1)} \; \|u\|_{H^k(\widehat{K})}.$$

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.