Serie 7

Theoretical Exercises: 1.–2.

Practical Exercise: 3.

1. (Legendre polynomials)

We define the Legendre Polynomials $\{L_n(x)\}_{n=0}^{+\infty}$ by the Rodriguez formula:

$$L_n(x) = \frac{1}{n!\,2^n}\Big[(x^2-1)^n\Big]^{(n)} \tag{1}$$

where $f^{(n)}$ denotes the n^{th}-derivative of f.

(a) Prove that:

$$L_0(x) = 1, \quad L_1(x) = x, \quad L_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \tag{2}$$

(b) For all $n \geqslant 1$, show that this holds true:

$$L_n(x) = \frac{L'_{n+1}(x) - L'_{n-1}(x)}{2n+1} \tag{3}$$

(c) Prove the orthogonality relation:

$$\begin{cases} \displaystyle\int_{-1}^{1} L_n(x)\,L_m(x)\,dx & = \dfrac{2}{2n+1} \qquad \text{for } n = m, \\ & = 0 \qquad \text{else.} \end{cases} \tag{4}$$

Hint: Use n integrations by part and the formula:

$$\int_{-1}^{1}(x^2-1)^n = (-1)^n \frac{\left(2^n n!\right)^2}{(2n)!}\frac{2}{2n+1}. \tag{5}$$

(d) For all $n \geqslant 1$, prove the following relation:

$$(n+1)\,L_{n+1}(x) + n\,L_{n-1}(x) = (2n+1)\,x\,L_n(x) \tag{6}$$

Hint: Show

$$x\, L_n(x) \;=\; L_{n+1}(x) \;-\; \frac{1}{(n-1)!\, 2^n} \left[(x^2-1)^n\right]^{(n-1)} \tag{7}$$

and

$$L_{n+1}(x) - L_{n-1}(x) \;=\; \frac{2n+1}{n!\, 2^n}\left[(x^2-1)^n\right]^{(n-1)} \tag{8}$$

Remark: An easy way to compute the Legendre polynomial is given by:

$$\left\{ \begin{array}{l} L_0(x) \;=\; 1, \quad L_1(x) \;=\; x, \\ L_{n+1}(x) \;=\; \dfrac{2n+1}{n+1}\, x\, L_n(x) \;-\; \dfrac{n}{n+1}\, L_{n-1}(x). \end{array} \right. \tag{9}$$

2. (1D-hierarchic shape functions for $\mathcal{S}_p^0$)

On the interval $[-1; 1]$, we define the following local shape functions $\{\widehat{b}_n\}_{n=1}^{p}$:

$$\left\{ \begin{array}{l} \widehat{b}_1(\xi) \;=\; \frac{1-\xi}{2}, \quad \widehat{b}_2(\xi) \;=\; \frac{1+\xi}{2}, \\ \widehat{b}_n(\xi) \;=\; \sqrt{\dfrac{2n-3}{2}} \displaystyle\int_{-1}^{\xi} L_{n-2}(t)\, dt, \quad \forall n \geqslant 3. \end{array} \right. \tag{10}$$

(a) Prove that the mass matrix

$$\widehat{M} \;=\; \left(\int_{-1}^{1} \widehat{b}_i(\xi)\, \widehat{b}_j(\xi)\, d\xi \right)_{i,j=1}^{p} \tag{11}$$

is given by:

$$\widehat{M} \;=\; \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{10}} & 0 & \cdots & \cdots & \cdots & 0 \\ & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{10}} & 0 & \cdots & \cdots & \cdots & 0 \\ & & \widehat{M}_{2,2} & 0 & \widehat{M}_{2,4} & \ddots & & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & & \widehat{M}_{i,i} & 0 & \widehat{M}_{i,i+2} & \ddots & \vdots \\ & & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & & \widehat{M}_{p-2,p-2} & 0 & \widehat{M}_{p-2,p} \\ & & & & & & & \widehat{M}_{p-1,p-1} & 0 \\ \text{sym} & & & & & & & & \widehat{M}_{p,p} \end{pmatrix} \tag{12}$$

with

$$\widehat{M}_{i,i} \;=\; \frac{2}{(2i-1)(2i-5)} \quad \text{and} \quad \widehat{M}_{i,i+2} \frac{-1}{(2i-1)\sqrt{(2i-3)(2i+1)}} \tag{13}$$

Hint: Prove that

$$\begin{cases} \widehat{b}_1(x) &= \dfrac{L_0(x) - L_1(x)}{2}, \quad \widehat{b}_1(x) \;=\; \dfrac{L_0(x) + L_1(x)}{2}, \\ \widehat{b}_n(x) &= \dfrac{1}{\sqrt{2(2n-3)}}\big(L_{n-1}(x) - L_{n-3}(x)\big), \end{cases} \tag{14}$$

and use the orthogonality relation (4).

(b) Show that the stiffness matrix

$$\widehat{K} \;=\; \Big(\int_{-1}^{1} \widehat{b}_i'(\xi)\, \widehat{b}_j'(\xi)\, d\xi\Big)_{i,j=1}^{P} \tag{15}$$

is given by:

$$\widehat{K} \;=\; \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ & & 1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ \text{sym} & & & & & 1 \end{pmatrix} \tag{16}$$

(c) On the interval $[-1; 1]$, we introduce the following mesh:

$$\mathcal{M} \;=\; \Big\{K_i\Big\}_{i=1}^{M} \tag{17}$$

with

$$K_i = \big[(i-1)h; ih\big],\;\; N \in \mathbb{N} \text{ and } h = \frac{1}{M}. \tag{18}$$

We define the following maps $\varphi_i : K_i \to [0; 1]$

$$\varphi_i(x) \;=\; -1 + 2 \cdot \frac{x - (i-1)h}{h}. \tag{19}$$

We define the following set of functions $\{b_N^i\}_{i=1}^{Mp+1}$:

- For $i = 0, \ldots, M$

$$\begin{cases} b_N^{ip+1}(x) &= \widehat{b}_2(\varphi_i(x)), \qquad \text{if } x \in K_i, \qquad (K_0 \in \emptyset) \\ &= \widehat{b}_1(\varphi_{i+1}(x)), \qquad \text{if } x \in K_{i+1}, \qquad (K_{M+1} \in \emptyset) \\ &= 0, \qquad \text{else.} \end{cases} \tag{20}$$

- For $i = 1, \ldots, M$ and $j = 2, \ldots, p$

$$\begin{cases} b_N^{(i-1)p+j}(x) &= \widehat{b}_{j+1}(\varphi_i(x)), \qquad \text{if } x \in K_i, \\ &= 0, \qquad \text{else.} \end{cases} \tag{21}$$

Show that $\{b_N^i\}_{i=1}^{Mp+1}$ is a basis of

$$\mathcal{S}_p^0(\mathcal{M}) \;=\; \left\{C^0([0;1] \,/\, u_{|K_i} \in \mathcal{P}_p(K_i))\right\} \tag{22}$$

(d) Compute the mass matrix

$$M_N \;=\; \left(\int_0^1 b_N^i(x)\, b_N^j(x)\, dx\right)_{i,j=1}^{Mp+1} \tag{23}$$

and the stiffness matrix

$$K_N \;=\; \left(\int_0^1 (b_N^i)'(x)\, (b_N^j)'(x)\, dx\right)_{i,j=1}^{Mp+1} \tag{24}$$

with respect to h, $\widehat{M}$ and $\widehat{K}$.

3. We consider $\Omega =]0;1[$, and $\mathcal{M} \;=\; \{](i-1)h, ih[\}_{i=1}^{M}$, with $N = 1/h$.

Write a MATLAB code which computes the $H^1(\Omega)$-orthogonal projection of $u^\lambda(x) = x^\lambda$ onto $\mathcal{S}_p^0(\mathcal{M})$ (with $\lambda > \frac{1}{2}$).

If $\mathsf{P}_h : H^1(\Omega) \to H^1(\Omega)$ is the $H^1(\Omega)$-orthogonal projection onto $\mathcal{S}_p^0(\mathcal{M})$, we recall that, for all $u \in H^1(\Omega)$, $\mathsf{P}_h u$ is characterized by:

$$(P_h u; v_h)_{H^1(\Omega)} \;=\; (u; v_h)_{H^1(\Omega)}, \quad \forall v_h \in \mathcal{S}_p^0(\mathcal{M}), \tag{25}$$

For $p = 2, 3, 4$, $\lambda = 2.5$ and $N = 1, 2, ..., 2^6$ compute

$$\|u - \mathsf{P}_h u\|_{H^1(\Omega)} \quad \text{and} \quad \|u - \mathsf{P}_h u\|_{L^2(\Omega)} \tag{26}$$

and draw the following curves in log – log scales (for p=2,3,4)

$$h \longrightarrow \|u - P_h u\|_{H^1(\Omega)} \quad \text{and} \quad h \longrightarrow \|u - P_h u\|_{L^2(\Omega)} \tag{27}$$

Hints:

- Choose the discreet space spanned with the 1D-hierarchic shape fucntions for $\mathcal{S}_p^0$ (see exercise 2)
- To compute the matrix associate to the left hand side of (25), use the analytic formula of exercise 2.
- To compute the right hand side use the Gauss-Legendre quadrature rule:

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http://www.sam.math.ethz.ch/~tordeux/gauleg.m

QUADRULE = GAULEG(A,B,N,TOL) computes
 the N-point Gauss-Legendre quadrature
 rule on the interval [A,B] up to the
 prescribed tolerance TOL. If no tolerance
 is prescribed GAULEG uses the machine
 precision EPS.

 Note that all quadrature rules obtained
 from GAULEG are of order 2*N-1.

 The struct QUADRULE contains the following fields:
 W N-by-1 matrix specifying the weights
   of the quadrature rule.
 X N-by-1 matrix specifying the abscissae of the
   quadrature rule.

   Example:

   QuadRule = gauleg(0,1,10,1e-6);
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Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.