Serie 6

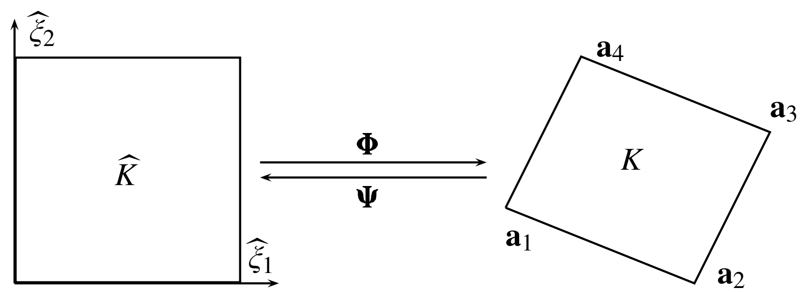
Theoretical Exercises: 1.–3.

Practical Exercise: none

1. Consider the following coordinate transformation $\mathbf{\Phi} : \widehat{K} \to K$ which maps the unit square $]0;1[^2$ to an arbitrary quadrangle with vertices $\{\mathbf{a}_i = (x_i, y_i)\}_{i=1}^4$

$$\mathbf{\Phi}(\widehat{\xi}_1, \widehat{\xi}_2) \quad = \quad \widehat{\xi}_1\widehat{\xi}_2\, \mathbf{a}_1 + \widehat{\xi}_1(1-\widehat{\xi}_2)\, \mathbf{a}_2 + (1-\widehat{\xi}_1)(1-\widehat{\xi}_2)\, \mathbf{a}_3 + (1-\widehat{\xi}_1)\widehat{\xi}_2\, \mathbf{a}_4$$

and the inverse transformation $\mathbf{\Psi} := \mathbf{\Phi}^{-1}$.



(i) Compute the Jacobian matrix $D\mathbf{\Phi}(\boldsymbol{\xi})$ and express its determinant $|\det D\mathbf{\Phi}(\boldsymbol{\xi})|$ with respect to $\widehat{\xi}_1, \widehat{\xi}_2$, $\{\det(\mathbf{a}_i - \mathbf{a}_{i-1}|\mathbf{a}_i - \mathbf{a}_{i+1})\}_{i=1}^4$ ($\mathbf{a}_0 = \mathbf{a}_4$ and $\mathbf{a}_5 = \mathbf{a}_1$)

(ii) Show that for an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $N \in \mathbb{N}$ following holds

$$\|\mathbf{A}\| \quad \leq \quad \|\mathbf{A}\|_F$$

where $\|.\|_F$ denotes the Frobenius norm on $\mathbb{R}^{N,N}$ which is defined by

$$\|\mathbf{A}\|_F = \left(\sum_{i,j=1}^{N} a_{ij}^2\right)^{\frac{1}{2}}, \qquad \mathbf{A} = \left(a_{ij}\right)_{i,j=1}^{N}.$$

(iii) Show that for an arbitrary $u \in H^1(K)$ following holds

$$|u|_{H^1(K)} \quad \leq \quad \max_{\text{¸}\in K} \|D\mathbf{\Psi}(\boldsymbol{\xi})\|_F \,|\det D\mathbf{\Psi}(\boldsymbol{\xi})|^{-\frac{1}{2}} |\widehat{u}|_{H^1(\widehat{K})}.$$

where $\widehat{u} \in H^1(\widehat{K})$ is defined by $\widehat{u}(\widehat{\boldsymbol{\xi}}) := u(\mathbf{\Phi}(\widehat{\boldsymbol{\xi}}))$.

(iv) Show that for an arbitrary $u \in H^2(K)$ there holds

$$\left(\sum_{k=1}^{2} \left\| \frac{\partial^2 \widehat{u}}{\partial \widehat{\xi}_k^2} \right\|^2_{L^2(\widehat{K})} \right)^{\frac{1}{2}} \leq \max_{\widehat{\xi} \in \widehat{K}} \| D\boldsymbol{\Phi}(\widehat{\boldsymbol{\xi}}) \|_F^2 | \det D\boldsymbol{\Phi}(\widehat{\boldsymbol{\xi}}) |^{-\frac{1}{2}} \| u \|_{H^2(K)}$$

where $\widehat{u} \in H^2(\widehat{K})$ is defined by $\widehat{u}(\widehat{\boldsymbol{\xi}}) := u(\boldsymbol{\Phi}(\widehat{\boldsymbol{\xi}}))$.

(vi) Let $\mathsf{Q} : H^1(K) \to H^1(K)$ be an operator satisfying:

$$\mathsf{Q}u = u, \quad \forall u \in \mathrm{Q}_1 \text{ such that } \widehat{u} \in \mathrm{Q}_1$$

Show that the inequality

$$\begin{cases} |u - \mathsf{Q}u|_{H^1(K)} \leq \gamma \max_{\xi \in K} \| D\boldsymbol{\Psi}(\boldsymbol{\xi}) \|_F | \det D\boldsymbol{\Psi}(\boldsymbol{\xi}) |^{-\frac{1}{2}} \\ \qquad \times \max_{\widehat{\xi} \in \widehat{K}} \| D\boldsymbol{\Phi}(\widehat{\boldsymbol{\xi}}) \|_F^2 | \det D\boldsymbol{\Phi}(\widehat{\boldsymbol{\xi}}) |^{-\frac{1}{2}} \| u \|_{H^2(K)} \end{cases}$$

holds for all $u \in H^2(K)$, where γ is a constant.

Hint: Use the Bramble-Hilbert Lemma:

If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz-domain and $m \in \mathbb{N}$, then $\exists\, \gamma = \gamma(m, \Omega) > 0$:

$$\inf_{p \in \mathrm{Q}_{m-1}(\Omega)} \| v - p \|_{H^m(\Omega)} \leq \gamma \left(\sum_{i=1}^{d} \left\| \frac{\partial^m v}{\partial \xi_i^m} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \quad \forall v \in H^m(\Omega) .$$

2. For Ω a bounded Lipschitz domain, then prove the following result:

$$\| v \|^2_{H^1(\Omega)} \leqslant C \, \| v \|_{L^2(\Omega)} \, \| v \|_{H^2(\Omega)}, \quad \forall v \in H^2(\Omega).$$

Hint: Prove the result for $\Omega = \mathbb{R}^2$ and use the extension theorem.

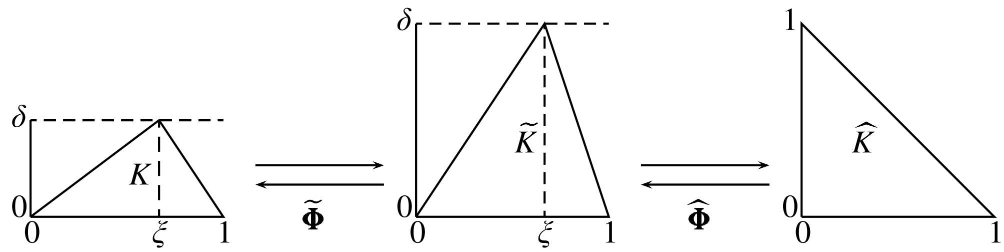
3. Let K be a triangle with vertices $(0,0)$, $(1,0)$, (ξ, δ), with $0 \leqslant \xi \leqslant 1$, $0 \leqslant \delta \leqslant 1$. Using the mapping

$$\mathbf{x} = \widetilde{\boldsymbol{\Phi}}(\widetilde{\mathbf{x}}) = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \widetilde{\mathbf{x}},$$

we define the function $\widetilde{u}(\widetilde{x}) = u(\widetilde{\boldsymbol{\Phi}}(\widetilde{\mathbf{x}}))$ which is defined on $\widetilde{K} = \widetilde{\boldsymbol{\Phi}}^{-1}(K)$. Then, using the mapping

$$\widetilde{\mathbf{x}} = \widehat{\boldsymbol{\Phi}}(\widehat{\mathbf{x}}) = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \widehat{\mathbf{x}} \tag{1}$$

we again define $\widehat{u}(\widehat{x}) = u(\widehat{\mathbf{\Phi}}(\widehat{\mathbf{x}}))$ which is defined on $\widehat{K} = \widehat{\mathbf{\Phi}}^{-1}(\widetilde{K})$.



(i) Prove that

$$\begin{cases} \|\widetilde{u}\|_{L^2(\widetilde{K})} & = \dfrac{1}{\sqrt{\delta}} \|u\|_{L^2(K)}, \\ \left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2}\right\|_{L^2(\widetilde{K})} & = \dfrac{1}{\sqrt{\delta}} \left\|\dfrac{\partial^2 u}{\partial x^2}\right\|_{L^2(K)}, \\ \left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}\partial \widetilde{y}}\right\|_{L^2(\widetilde{K})} & = \sqrt{\delta} \left\|\dfrac{\partial^2 u}{\partial x \partial y}\right\|_{L^2(K)}, \\ \left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{y}^2}\right\|_{L^2(\widetilde{K})} & = \sqrt{\delta^3} \left\|\dfrac{\partial^2 u}{\partial y^2}\right\|_{L^2(K)}. \end{cases} \tag{2}$$

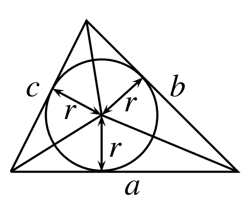
(ii) Use the shape regularity to prove that $\exists \underline{C},\ \overline{C} > 0$ such that:

$$\begin{cases} \underline{C}\, \|\widetilde{u}\|_{L^2(\widetilde{K})} & \leqslant \|\widehat{u}\|_{L^2(\widehat{K})} & \leqslant \overline{C}\, \|\widetilde{u}\|_{L^2(\widetilde{K})}, \\ \underline{C}\left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2}\right\|_{L^2(\widetilde{K})} & \leqslant \left\|\dfrac{\partial^2 \widehat{u}}{\partial \widehat{x}^2}\right\|_{L^2(\widehat{K})} & \leqslant \overline{C} \left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2}\right\|_{L^2(\widetilde{K})}, \\ \underline{C}\left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}\partial \widetilde{y}}\right\|_{L^2(\widetilde{K})} & \leqslant \left\|\dfrac{\partial^2 \widehat{u}}{\partial \widehat{x}\partial \widehat{y}}\right\|_{L^2(\widehat{K})} & \leqslant \overline{C}\left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{x}\partial \widetilde{y}}\right\|_{L^2(\widetilde{K})}, \\ \underline{C}\left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{y}^2}\right\|_{L^2(\widetilde{K})} & \leqslant \left\|\dfrac{\partial^2 \widehat{u}}{\partial \widehat{y}^2}\right\|_{L^2(K)} & \leqslant \overline{C}\left\|\dfrac{\partial^2 \widetilde{u}}{\partial \widetilde{y}^2}\right\|_{L^2(\widetilde{K})}. \end{cases}$$

Hint: Use the formula

$$r(a+b+c) = 2|K| \tag{3}$$

where r is the radius of the inscribed circle and a, b and c are the lengths of the sides.



(iii) Prove that

$$\exists C > 0, \quad \|u - \mathsf{I}_{1,K} u\|_{L^2(K)} \leqslant C\ |u|_{H^2(K)}. \tag{4}$$

(iv) For a triangular mesh M over Ω, show that:

$$\exists C > 0, \quad \|u - \mathsf{I}_{1,\mathrm{M}} u\|_{L^2(\Omega)} \leqslant C\ h_{\mathrm{M}}^2\ |u|_{H^2(\Omega)}, \tag{5}$$

where $C > 0$ does not depend on the mesh regularity.

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.