Serie 5

Context: Galerkin approximation.

Theoretical Exercises: 1.–5.

Practical Exercise: none

1. What is the dimension of the space $\mathcal{P}_p(K) \cap H_0^1(K)$ for a triangle $K \subset \mathbb{R}^2$ and $p \in \mathbb{N}$? For the simplest non-trivial case describe a basis of this space.

2. For a triangle $K \subset \mathbb{R}^2$ find a basis of the space $\{\mathbf{u} \in (\mathcal{P}_1(K))^2 : \operatorname{div}\mathbf{u} \in \mathcal{P}_0(K)\}$.

3. (2D-Hermitian finite elements)

Let Ω be the unit square and $\partial\Omega$ its boundary:

$$\Omega =]0;1[^2, \quad \partial\Omega = \overline{\Omega} \setminus \Omega \tag{1}$$

We consider the problem:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that:} \\ -\Delta u = f, \quad \text{in } \Omega, \quad \text{with } f \in L^2(\Omega). \end{cases} \tag{2}$$

(a) Write the variational formulation

$$\text{Find } u \in V \text{ such that } \mathsf{a}(u,v) = \mathsf{f}(v), \quad \forall v \in V, \tag{3}$$

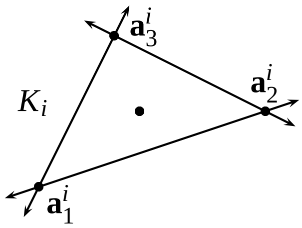
and show that problem (2) has a unique solution.

(b) For $V_N \subset H_0^1(\Omega)$, show that the following problem has a unique solution:

$$\text{Find } u_N \in V_N \text{ such that } \mathsf{a}(u_N, v_N) = \mathsf{f}(v_N), \quad \forall v_N \in V_N. \tag{4}$$

(c) We define the tringulation $\mathcal{M} = \left\{K_i\right\}_{i=1}^{M}$:

$$K_i = \{\mathbf{x} \in \mathbb{R}^2 \ / \ \sum_{j=1}^{3} \lambda_j \ a_j^i \text{ with } \sum_{j=1}^{3} \lambda_j = 1 \text{ and } \lambda_j \in [0;1] \text{ for } j = 1,2,3\}. \tag{5}$$



We introduce the following local discreet space:

$$V_N(K) = \mathcal{P}_3(K), \quad \forall K \in \mathcal{M}. \tag{6}$$

We consider the system of linear forms of V_{K_i} $\mathcal{D}_{K_i} = \{\beta_{K_i}^j\}_{j=1}^{10}$ defined by:

$$\left\{ \begin{array}{lcl} \beta_{K_i}^1(v) & = & v(\mathbf{a}_i^1), \\ \beta_{K_i}^2(v) & = & v(\mathbf{a}_i^2), \\ \beta_{K_i}^3(v) & = & v(\mathbf{a}_i^3), \\ \beta_{K_i}^4(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^1) \cdot (\mathbf{a}_i^2 - \mathbf{a}_i^1), \\ \beta_{K_i}^5(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^2) \cdot (\mathbf{a}_i^3 - \mathbf{a}_i^2), \\ \beta_{K_i}^6(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^3) \cdot (\mathbf{a}_i^1 - \mathbf{a}_i^3), \\ \beta_{K_i}^7(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^1) \cdot (\mathbf{a}_i^3 - \mathbf{a}_i^1), \\ \beta_{K_i}^8(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^2) \cdot (\mathbf{a}_i^1 - \mathbf{a}_i^2), \\ \beta_{K_i}^9(v) & = & \mathbf{grad}\, v_{|K_i}(\mathbf{a}_i^3) \cdot (\mathbf{a}_i^2 - \mathbf{a}_i^3), \\ \beta_{K_i}^{10}(v) & = & v(\dfrac{\mathbf{a}_i^1 + \mathbf{a}_i^2 + \mathbf{a}_i^3}{3}). \end{array} \right. \tag{7}$$

Show that $\mathcal{D}_{K_i}$ is unisolvent.

(d) Find the local shape functions $\{b_{K_i}^j\}_{j=1}^{10}$ associated to the local degrees of freedom $\mathcal{D}_{K_i}$. Write them using the barycentric coordinates. Which of them are associated with vertices, edges, the interior of a cell?

(e) Show that matching local degrees of freedom at a node ensures

$$v_N \in V_N = \{v_N \in C^0(\Omega) \ / \ v_{N|K} \in V_N(K), \ \forall K \in \mathcal{M}\}. \tag{8}$$

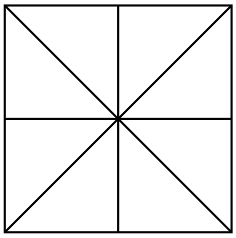
How can we ensure $v_{N|\partial\Omega} \equiv 0$?

(f) What are the global degrees of freedom arising from the above local degrees of freedom?

(g) Compute the local stiffness matrix $\mathbf{S}$:

$$\mathbf{S}_{K_i}^{j,k} \ = \int_{K_i} \mathbf{grad}\, b_{K_i}^j(x)\ \mathbf{grad}\, b_{K_i}^k(x)\ dx. \tag{9}$$

(h) What is the dimension of V_N for the following mesh?



Give a sharp bound for the number of nonzero elements of the global stiffness matrix.

4. Let Ω be the Torus

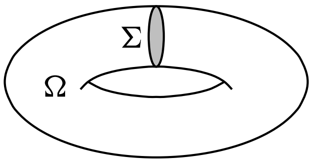
$$\Omega \;=\; \{\mathbf{x} \in \mathbb{R}^3 \;/\; \exists \mathbf{y} \in \mathcal{C} \;\|x-y\|_2 < \frac{1}{2}\} \tag{10}$$

with

$$\mathcal{C} \;=\; \{\mathbf{y} \in \mathbb{R}^3 \;/ (y_1)^2 + (y_2)^2 \;=\; 1 \quad \text{and} \;\; y_3 = 0\}. \tag{11}$$

We define the cut Σ

$$\Sigma \;=\; \{\mathbf{y} \in \Omega \;/\; \mathbf{y}_1 = 0 \text{ and } \mathbf{y}_2 > 0\} \tag{12}$$



Find the boundary value problem associated with the variational formulation:

$$\left\{\begin{array}{l} \text{Find } u \in V \text{ such that for all } v \in V, \\ \displaystyle\int_\Omega \nabla u \;\nabla v \; d\mathbf{x} \;=\; \int_\Sigma [v] \; ds \end{array}\right. \tag{13}$$

where

$$\left\{\begin{array}{l} [v] \;=\; v(0^+, \mathbf{y}_2, \mathbf{y}_3) \;-\; v(0^-, \mathbf{y}_2, \mathbf{y}_3), \\ V = \left\{u \in H^1(\Omega \setminus \Sigma) \;\big/\; \int_\Omega u \; d\mathbf{x} = 0\right\}. \end{array}\right. \tag{14}$$

5. (p-hierarchical basis of $\mathcal{S}_2^0(\mathcal{M})$)

We consider a domain Ω and a triangular mesh $\mathcal{M} = \{K_i\}_{i=1}^{M}$.

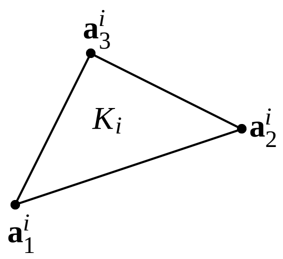
In each K_i, we introduce the discreet space $\mathcal{P}_2(K_i)$ and define the local degrees of freedom $\{\beta_{K_i}^{j}\}_{j=1}^{6}$:

$$\beta_{K_i}^{1}(v) \;=\; v(\mathbf{a}_i^1), \quad \beta_{K_i}^{2}(v) \;=\; v(\mathbf{a}_i^2), \quad \beta_{K_i}^{3}(v) \;=\; v(\mathbf{a}_i^3), \tag{15}$$

$$\beta_{K_i}^{3+j}(v) \;=\; \int_{-1}^{1} (\frac{3}{2}\xi^2 - \frac{1}{2})\; v(\varphi_{K_I}^{j}(\xi))\; d\xi, \quad j = 1, 2, 3. \tag{16}$$

with $\varphi_{K_i}^{j} : [-1; 1] \to \mathbb{R}^2$ given by:

$$\left\{\begin{array}{rcl} \varphi_{K_i}^{1}(\xi) &=& \dfrac{1-\xi}{2}\; \mathbf{a}_{K_i}^{2} \;+\; \dfrac{1+\xi}{2}\; \mathbf{a}_{K_i}^{3}, \\ \varphi_{K_i}^{2}(\xi) &=& \dfrac{1-\xi}{2}\; \mathbf{a}_{K_i}^{3} \;+\; \dfrac{1+\xi}{2}\; \mathbf{a}_{K_i}^{1}, \\ \varphi_{K_i}^{3}(\xi) &=& \dfrac{1-\xi}{2}\; \mathbf{a}_{K_i}^{1} \;+\; \dfrac{1+\xi}{2}\; \mathbf{a}_{K_i}^{2}. \end{array}\right. \tag{17}$$



(a) Compute the local degrees of freedom (in terms of barycentric coordinates).

(b) Show the trace fixing property of the local degree of freedom.

(c) Show that the resulting global finite elements space is $\mathcal{S}_2^0(\mathcal{M})$.

Remark. This demonstrates that there are several options for choosing local shape functions for $\mathcal{S}_p^0(\mathcal{M})$.

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.