Serie 4

Context: Galerkin approximation.

Theoretical Exercises: 1.–2.

Practical Exercise: 3.

1. (1D-Hermitian finite elements)

We consider the problem:

$$\begin{cases} \text{Find } u \in H^1(]0;1[) \text{ such that:} \\ -u'' + u \;=\; f, \quad \text{with } f \in L^2(]0;1[), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1}$$

(a) Write the variational formulation

$$\text{Find } u \in V \text{ such that } \mathsf{a}(u, v) \;=\; \mathsf{f}(v), \quad \forall v \in V, \tag{2}$$

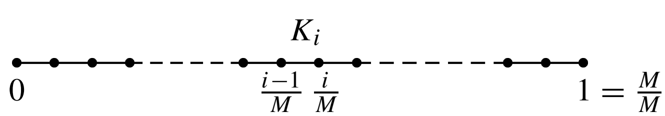
and show that problem (1) has a unique solution.

(b) For $V_N \subset H^1([0;1])$, show that the following problem has a unique solution:

$$\text{Find } u_N \in V_N \text{ such that } \mathsf{a}(u_N, v_N) \;=\; \mathsf{f}(v_N), \quad \forall v_N \in V_N. \tag{3}$$

(c) We define the mesh $\mathcal{M}$:

$$\mathcal{M} \;=\; \left\{K_i\right\}_{i=1}^{M} \quad \text{with } K_i \;=\; \left]\frac{i-1}{M}; \frac{i}{M}\right[. \tag{4}$$



We introduce the following local discreet space:

$$V_N(K) = \mathcal{P}_3(K), \quad \forall K \in \mathcal{M}. \tag{5}$$

We consider the system of linear forms of V_{K_i} $\mathcal{D}_{K_i} = \{\beta^1_{K_i}, \beta^2_{K_i}, \beta^3_{K_i}, \beta^4_{K_i}\}$ defined by:

$$\begin{cases} \beta^1_{K_i}(v) &=& v(\dfrac{i-1}{n}), \\ \beta^2_{K_i}(v) &=& \dfrac{1}{N} \lim\limits_{x \to 0^+} v'(\dfrac{i-1}{N} + x), \\ \beta^3_{K_i}(v) &=& v(\dfrac{i}{n}), \\ \beta^4_{K_i}(v) &=& \dfrac{1}{N} \lim\limits_{x \to 0^-} v'(\dfrac{i}{N} + x). \end{cases} \tag{6}$$

Show that $\mathcal{D}_{K_i}$ is unisolvent ($\mathcal{D}_{K_i}$ is a basis of $(V_{K_i})'$).

(d) Find the local shape functions $\{b^1_{K_i}, b^2_{K_i}, b^3_{K_i}, b^4_{K_i}\}$ associated to the local degrees of freedom $\mathcal{D}_{K_i}$.

Hint: Find $b^l_{K_i} \in \mathcal{P}_3(K_i)$ such that $\beta^k_{K_i}(b^l_{K_i}) = \delta_{k,l}$, for all $k, l = 1, ..., 4$.

(e) Show that matching local degrees of freedom at a node ensures

$$v_N \in V_N = \{v_N \in C^1([0;1]) \ / \ v_{N|K} \in V_N(K), \ \forall K \in \mathcal{M}\}. \tag{7}$$

(f) Prove that the global degrees of freedom $\{b^j_N\}_{j=1}^{2N}$ arising from the above local degrees of freedom are:

$$\beta^{2i}_N(v_N) = v_N(\frac{i}{N}), \ \ \beta^{2i+1}_N(v_N) = \frac{1}{N} \ {v_N}'(\frac{i}{N}), \quad \forall i = 0, ..., N. \tag{8}$$

What are the global shape functions $\{b^i_N\}_{i=1}^{2N+2}$?

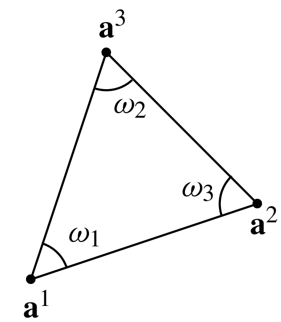
(g) Compute the stiffness matrix $\mathbf{S}$:

$$\mathbf{S}_{j,i} \ = \ \int_0^1 (b^i_N)'(x) \ (b^j_N)'(x) \ dx \tag{9}$$

and the mass matrix $\mathbf{M}$

$$\mathbf{M}_{j,i} \ = \ \int_0^1 b^i_N(x) \ b^j_N(x) \ dx. \tag{10}$$

2. Let $K \subset \mathbb{R}^2$ be a triangle with the vertices $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$, then we denote by $\omega_1, \omega_2, \omega_3$ its inner angles.



$$\mathbf{a}^1 := \begin{pmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^1 \end{pmatrix}, \quad \mathbf{a}^2 := \begin{pmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^2 \end{pmatrix}, \quad \mathbf{a}^3 := \begin{pmatrix} \mathbf{a}_1^3 \\ \mathbf{a}_2^3 \end{pmatrix}. \tag{11}$$

The barycentric coordinates $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})$ at the point $\mathbf{x} \in \mathbb{R}^2$, which are given by

$$\begin{aligned}
\lambda_1(\mathbf{x}) &:= \frac{1}{2|K|} \begin{pmatrix} \mathbf{x}_1 - \mathbf{a}_1^2 \\ \mathbf{x}_2 - \mathbf{a}_2^2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_2^2 - \mathbf{a}_2^3 \\ \mathbf{a}_1^3 - \mathbf{a}_1^2 \end{pmatrix}, \\
\lambda_2(\mathbf{x}) &:= \frac{1}{2|K|} \begin{pmatrix} \mathbf{x}_1 - \mathbf{a}_1^3 \\ \mathbf{x}_2 - \mathbf{a}_2^3 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_2^3 - \mathbf{a}_2^1 \\ \mathbf{a}_1^1 - \mathbf{a}_1^3 \end{pmatrix}, \\
\lambda_3(\mathbf{x}) &:= \frac{1}{2|K|} \begin{pmatrix} \mathbf{x}_1 - \mathbf{a}_1^1 \\ \mathbf{x}_2 - \mathbf{a}_2^1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_2^1 - \mathbf{a}_2^2 \\ \mathbf{a}_1^2 - \mathbf{a}_1^1 \end{pmatrix},
\end{aligned}$$

can be used to define the element shape functions $b_1(\mathbf{x}), b_2(\mathbf{x}), b_3(\mathbf{x})$ for piecewise linear finite elements

$$b_K^1(\mathbf{x}) := \lambda_1(\mathbf{x}), \quad b_K^2(\mathbf{x}) := \lambda_2(\mathbf{x}), \quad b_K^3(\mathbf{x}) := \lambda_3(\mathbf{x}), \tag{12}$$

on each element. For piecewise linear finite elements the element stiffness matrix $\mathbf{A}_K$ for the Laplacian is given by

$$\mathbf{A}_K^{ij} := \int_K \nabla b_K^i \cdot \nabla b_K^j \quad (1 \leq i, j \leq 3).$$

(a) Show that the sum over all entries of a row or a column of the local stiffness matrix equals zero.

(b) Show that the following equations hold true

$$\int_K \nabla b_K^1 \cdot \nabla b_K^2 = -\frac{1}{2}\, cotan(\omega_3)\,,$$

$$\int_K \nabla b_K^1 \cdot \nabla b_K^3 = -\frac{1}{2}\, cotan(\omega_2)\,,$$

$$\int_K \nabla\, b_K^2 \cdot \nabla b_K^3 = -\frac{1}{2}\, cotan(\omega_1)\,.$$

(c) Show that the following equation holds true

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} \alpha_2+\alpha_3 & -\alpha_3 & -\alpha_2 \\ & \alpha_3+\alpha_1 & -\alpha_1 \\ \text{sym.} & & \alpha_1+\alpha_2 \end{bmatrix}$$

where $\alpha_1 := cotan(\omega_1)$, $\alpha_2 := cotan(\omega_2)$ and $\alpha_3 := cotan(\omega_3)$.

3. On the square

$$\Omega = \{(x, y) \in \mathbb{R}^2 \;/\; 0 \leqslant x \leqslant 1,\; 0 \leqslant y \leqslant 1\}, \tag{13}$$

we consider the problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \nabla u \nabla v = \int_\Omega f\; v, \end{cases} \tag{14}$$

with

$$f(x, y) = 13\pi^2\; \sin(3\pi x)\; \sin(2\pi y). \tag{15}$$

To compute a numerical approximation, we introduce the following discreet space:

$$\mathcal{S}_{2,0}^0(\mathcal{M}) = \{v \in C^0(\overline{\Omega}) \;/\; v_{|\partial\Omega} \equiv 0, \quad v_{|K} \in \mathcal{P}_2(K) \quad \forall K \in \mathcal{M}\}. \tag{16}$$

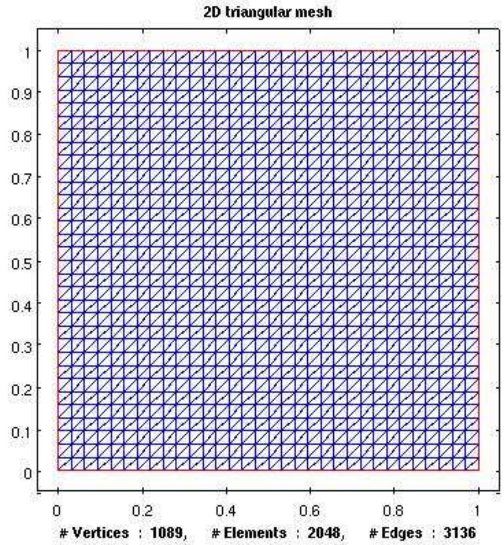
where $\mathcal{M}$ is a triangular mesh of Ω (see the following figure for an example)

The numerical approximation u_N of the exact solution u is the unique solution of the following problem:

$$\begin{cases} \text{Find } u_N \in \mathcal{S}_{2,0}^0(\mathcal{M}) \text{ such that:} \\ \displaystyle\int_\Omega \nabla u_N \cdot \nabla v_N = \int_\Omega f\; v_N, \quad \forall v_N \in \mathcal{S}_{2,0}^0(\mathcal{M}). \end{cases} \tag{17}$$

Write a MATLAB code which computes and plots the approximate numerical solution.

One can refer to the appendix for more informations.



Practical exercise: the solution is due on Tuesday January 9.

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.

Appendix: Quadratic finite elements

Linear finite elements you can find at the following adress MATLAB files which solves the same problem with $V_N(K) = \mathcal{S}^0_{1,0}(K)$:

http://www.sam.math.ethz.ch/ tordeux/LFE

mainLFE : computes and draws the solution

assemMat_LFE : assembles the galerkin matrix

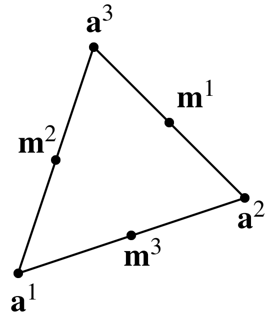
STIMA_Lapl_LFE : computes the local stiffness matrix

assemLoad_LFE : assembles the right hand side

assemDir_LFE : takes into account the Dirichlet boundary condition

plot_LFE : plots a (linear by element) function

Local shape function. Let K be an element of $\mathcal{M}$, $\{\mathbf{a}^i\}_{i=1}^3$ be its vertices and $\{\mathbf{m}^i\}_{i=1}^3$ be the midpoints of the edges.



We define the six local degrees of freedom (the β^i are forms of $\mathcal{P}_3(K)$)

$$\left\{ \begin{array}{lll} \beta_K^1(v) = v(\mathbf{a}_K^1), & \beta_K^2(v) = v(\mathbf{a}_K^2), & \beta_K^3(v) = v(\mathbf{a}_K^3), \\ \beta_K^4(v) = v(\mathbf{m}_K^1), & \beta_K^5(v) = v(\mathbf{m}_K^2), & \beta_K^6(v) = v(\mathbf{m}_K^3). \end{array} \right. \tag{18}$$

A nice way to express the local shape functions associated to these degrees of freedom consists in introducing the barycentric coordinates $\{\lambda_i\}_{i=1}^3$ uniquely defined by the relations:

$$\mathbf{x} \;=\; \sum_{i=1}^{3} \lambda_i(\mathbf{x})\ \mathbf{a}_K^i \quad \text{with} \quad \sum_{i=1}^{3} \lambda_i(\mathbf{x}) = 1. \tag{19}$$

The local shape functions are then given by:

$$\left\{ \begin{array}{lcl} b_K^1(\mathbf{x}) &=& \lambda_1(\mathbf{x})\ (2\lambda_1(\mathbf{x}) - 1), \\ b_K^2(\mathbf{x}) &=& \lambda_2(\mathbf{x})\ (2\lambda_2(\mathbf{x}) - 1), \\ b_K^3(\mathbf{x}) &=& \lambda_3(\mathbf{x})\ (2\lambda_3(\mathbf{x}) - 1), \\ b_K^4(\mathbf{x}) &=& 4\ \lambda_2(\mathbf{x})\ \lambda_3(\mathbf{x}), \\ b_K^5(\mathbf{x}) &=& 4\ \lambda_1(\mathbf{x})\ \lambda_3(\mathbf{x}), \\ b_K^6(\mathbf{x}) &=& 4\ \lambda_1(\mathbf{x})\ \lambda_2(\mathbf{x}). \end{array} \right. \tag{20}$$

The local stiffness matrix $A_K \in \mathbb{R}^{6\times 6}$ is the following:

$$A_K^{i,j} \;=\; \int_K \nabla b_K^i\ \nabla b_K^j. \tag{21}$$

Hint: to compute the local stiffness matrix, one can use the following formula:

$$\int_K \lambda_1^{\alpha_1}\ \lambda_2^{\alpha_2}\ \lambda_3^{\alpha_3} \;=\; |K|\ \frac{2\ \alpha_1!\ \alpha_2!\ \alpha_3!}{(\alpha_1+\alpha_2+\alpha_3+2)!}. \tag{22}$$