Serie 3

Context: Galerkin approximation.

Theoretical Exercises: 1.–5.

Practical Exercise: none

1. We consider the equation

$$\operatorname{div}(\mathbf{C}\,\mathbf{grad}(u)) = f, \quad \text{on }]0;1[^2, \quad \text{with } \mathbf{C} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

Find a linear transformation $\Phi : \widehat{\Omega} \to]0;1[^2$ (of the form $\Phi(\widehat{x}) = \mathbf{T}\widehat{x}$ with $\mathbf{T} \in \mathbb{R}^{2,2}$) which transforms (1) into

$$\Delta \Phi^* u = \Phi^* f? \tag{2}$$

What is $\widehat{\Omega}$?

2. How does the PDE:

$$\mathbf{grad}(u) \cdot \mathbf{b} = f \tag{3}$$

transforms under the change of variables $\Phi : \widehat{\Omega} \to \Omega$

3. Show that for $\mathbf{q} \in \left(C^1(\overline{\Omega})\right)^3$ and Ω a Lipschitz domain we have:

$$\int_\Omega \mathbf{curl}\,\mathbf{q} = \int_{\partial\Omega} \mathbf{n} \times \mathbf{q}, \tag{4}$$

with

$$\mathbf{curl}\,\mathbf{q} = \begin{bmatrix} \frac{\partial \mathbf{q_3}}{\partial x_2} - \frac{\partial \mathbf{q_2}}{\partial x_3} \\ \frac{\partial \mathbf{q_1}}{\partial x_3} - \frac{\partial \mathbf{q_3}}{\partial x_1} \\ \frac{\partial \mathbf{q_2}}{\partial x_1} - \frac{\partial \mathbf{q_1}}{\partial x_2} \end{bmatrix}. \tag{5}$$

Bitte wenden!

Hint: Use Gauss formula and the following identity for all constant vector field $\mathbf{v}$

$$\operatorname{div}(\mathbf{q} \times \mathbf{v}) \;=\; \textbf{curl}\,(\mathbf{q}) \cdot \mathbf{v} \;-\; \textbf{curl}\,(\mathbf{v}) \cdot \mathbf{q}. \tag{6}$$

Indeed, one has:

$$\left\{ \begin{array}{rcl} \operatorname{div}(\mathbf{q} \times \mathbf{v}) & = & \operatorname{div} \begin{bmatrix} \mathbf{q}_2\mathbf{v}_3 - \mathbf{q}_3\mathbf{v}_2 \\ \mathbf{q}_3\mathbf{v}_1 - \mathbf{q}_1\mathbf{v}_3 \\ \mathbf{q}_1\mathbf{v}_2 - \mathbf{q}_2\mathbf{v}_1 \end{bmatrix} \\ & = & +\left[\dfrac{\partial \mathbf{q}_3}{\partial x_2} - \dfrac{\partial \mathbf{q}_2}{\partial x_3}\right]\mathbf{v}_1 + \left[\dfrac{\partial \mathbf{q}_1}{\partial x_3} - \dfrac{\partial \mathbf{q}_3}{\partial x_1}\right]\mathbf{v}_2 + \left[\dfrac{\partial \mathbf{q}_2}{\partial x_1} - \dfrac{\partial \mathbf{q}_1}{\partial x_2}\right]\mathbf{v}_3 \\ & & -\left[\dfrac{\partial \mathbf{v}_3}{\partial x_2} - \dfrac{\partial \mathbf{v}_2}{\partial x_3}\right]\mathbf{q}_1 - \left[\dfrac{\partial \mathbf{v}_1}{\partial x_3} - \dfrac{\partial \mathbf{v}_3}{\partial x_1}\right]\mathbf{q}_2 - \left[\dfrac{\partial \mathbf{v}_2}{\partial x_1} - \dfrac{\partial \mathbf{v}_1}{\partial x_2}\right]\mathbf{q}_3 \\ & = & \textbf{curl}\,(\mathbf{q}) \cdot \mathbf{v} \;-\; \textbf{curl}\,(\mathbf{v}) \cdot \mathbf{q}. \end{array} \right.$$

4. We consider the following model for diffusion of a crystal which dissolves in a liquid phase and is submitted to a chemical reaction:

Consider functions:

- $u: \ \Omega \to \mathbb{R}$ (a concentration)
- $\mathbf{j}: \Omega \to \mathbb{R}^3$ (a flux)

satisfying the Fick's law (c: $\Omega \to \mathbb{R}$ is the diffusion coefficient, $c \in L^\infty(\Omega)$ and is uniformly positive)

$$\mathbf{j}(x) \;=\; -c(x)\,\textbf{grad}(u), \quad \forall x \in \Omega, \tag{7}$$

and the conservation of species with a reaction term ($q : \Omega \to \mathbb{R} \in L^\infty(\Omega)$ and is uniformly positive)

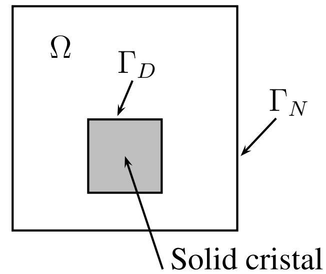
$$-\int_V q(x)\; u^2(x)\; dx \;=\; \int_{\partial V} \mathbf{j} \cdot \mathbf{n}\; ds, \quad \text{for all volumes } V \subset \Omega, \tag{8}$$

As no molecule can go outside the container, we consider Neumann boundary condition on Γ_N:

$$\mathbf{j} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N, \tag{9}$$

When the crystal is solid, its concentration is 1, Hence we put on Γ_D the inhomogeneous Dirichlet boundary condition.

$$\mathbf{u} = 1 \quad \text{on } \Gamma_D. \tag{10}$$



(i) Formulate the (non linear) boundary value problem (BVP) for u.
(ii) Write the (non linear) variational formulation associated to this BVP.

5. Show that the following proposition is not true:

For $\Omega \subset \mathbb{R}^2$ Lipschitz, $u \in H^{1/2}(\partial\Omega) \implies u \in C^0(\partial\Omega)$

Hint: use trace theorem, the function defined in polar coordinates $f(r, \varphi) = \log|\log r|$ on the domain $\{0 < r < 1/2,\ 0 < \varphi < \pi/2\}$.

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.