Serie 2

Context: Galerkin approximation.

Theoretical Exercises: 1.–5.

Practical Exercise: 6.

Let V_N and W_N be two finite dimensional spaces with $\dim(V_N) = N$ and $\dim(W_N) = P$. Let $A_N : V_N \to (W_N)'$ be a linear operator. Let f_N be an element of $(W_N)'$

$$\text{Find } u_N \in V_N, \quad \text{such that} \quad A_N u_N = f_N. \tag{1}$$

We introduce a basis $\mathfrak{B}_V = \{p_N^1, ..., p_N^N\}$ (resp. $\mathfrak{B}_W = \{q_N^1, ..., q_N^P\}$) of V_N (resp. W_N). Finally, we consider the problem:

$$\text{Find } \mathbf{u}_N \in \mathbb{C}^N \quad \mathbf{A}_N \mathbf{u}_N \;=\; \mathbf{f}_N, \tag{2}$$

with

$$\left\{ \begin{array}{lcl} \mathbf{A}_N & = & \left(\left\langle A_N p_N^i ; q_N^j \right\rangle_{W_N} \right)_{j,i=1}^{P,N} \in \mathbb{C}^{P \times N}, \\ \mathbf{f}_N & = & \left(\left\langle f ; q_N^j \right\rangle_{W_N} \right)_{j=1}^{P} \in \mathbb{C}^{P}, \\ u_N & = & \displaystyle\sum_{i=1}^{N} \mathbf{u}_N^i \, p_N^i. \end{array} \right. \tag{3}$$

1. (a) Show that problem (1) is equivalent to problem (2).

(b) Prove that:

$$\text{the problem (1) has generically } (\forall f) \text{ one solution} \quad \Longrightarrow \quad N \;=\; P. \tag{4}$$

Hint: Use that $\dim(Rg(A_N)) \;+\; dim(Ker(A_N)) \;=\; N$.

2. We set $N = P$, $V_N = W_N$ and $\mathfrak{B}_V = \mathfrak{B}_W$.

Compute the new Galerkin matrix $\underline{\mathbf{A}}_N$ and the right hand side vector $\underline{f}_N$ for problem (1) after a permutation of the basis vectors of $\mathfrak{B}_V$.

Hint: Introduce a permutation matrix $\mathbf{P}$.

Bitte wenden!

3. We set $V_N = W_N$ and $\mathfrak{B}_V = \mathfrak{B}_W$. We denote by $\mathfrak{B}_V^2$ the basis obtained by scaling the basis vectors of $\mathfrak{B}_V$:

$$\mathfrak{B}_V^2 \;=\; (\alpha_N^i p_N^i)_{i=1}^N \quad \text{with } \forall i = 1, ..., N \quad \alpha_N^i \in \mathbb{R}^*. \tag{5}$$

Express $\mathbf{A}_N^2$ and $\mathbf{f}_N^2$ with respect to $\mathbf{A}_N$, $\mathbf{f}_N$, α_N^i, where $\mathbf{A}_N^2$ and $\mathbf{f}_N^2$ are the Galerkin matrix associated to problem (1) and to the basis $\mathfrak{B}_V^2$

4. The operator equation (1) is set in finite dimensional spaces. Hence, after choosing bases for V_N and $(W_N)'$ it can be converted into a linear system of equations. Which bases have to be chosen, such that we exactly end up with (2)?

5. We set $N = P$, and we suppose that A_N is bijective:

For a fixed basis $\mathfrak{B}_W = (q_N^i)_{i=1}^N$ of W_N, define a basis $\mathfrak{B}_V = (p_N^i)_{i=1}^N$ of V_N such that:

$$u_N \;=\; \sum_{i=1}^{N} \left\langle f_N ; q_N^i \right\rangle \; p_N^i \tag{6}$$

where u_N is the solution of problem (1). What is the Galerkin matrix $\mathbf{A}_N$ associated to $\mathfrak{B}_V$ and $\mathfrak{B}_W$?

6. Consider the bilinear form:

$$\left\{ \begin{array}{llcl} \mathsf{a}: & L^2([0;1]) \times L^2([0;1]) & \rightarrow & \mathbb{R} \\ & (u,v) & \mapsto & \displaystyle\int_0^1 u(x)\, v(x). \end{array} \right. \tag{7}$$

(a) Does this bilinear form satisfy the inf-sup condition in $L^2([0;1])$?

(b) We set:

$$V_N \;=\; \text{span}(\mathfrak{B}_V) \subset L^2([0;1]) \quad \text{and} \quad W_N \;=\; \text{span}(\mathfrak{B}_W) \subset W \subset L^2([0;1]), \tag{8}$$

with:

$$\left\{ \begin{array}{lcl} \mathfrak{B}_V & = & \left\{ p_N^k : \; [0;1] \rightarrow \mathbb{R}, \; x \mapsto x^k \;\Big/\; k = 0, ..., N-1 \right\}, \\ \mathfrak{B}_W & = & \left\{ q_N^k : \; [0;1] \rightarrow \mathbb{R}, \; x \mapsto \mathbb{I}_{[\frac{k}{n}, \frac{k+1}{n}]}(x) \;\Big/\; k = 0, ..., N-1 \right\}, \end{array} \right. \tag{9}$$

where $\mathbb{I}_I(x) = 1$ if $x \in I$ and 0 else.

Compute the Galerkin matrix $\mathbf{A}_N$.

(c) Write a MATLAB code which computes the inf-sup constant γ_N for some N's.

Siehe nächstes Blatt!

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoritical exercises and 50% practical exercises (MATLAB) should be solved.

Appendix: Singular Value Decomposition (SVD):

Let $\mathbf{A}$ be an element of $\mathbb{C}^{N\times N}$. We will show that there exist $\mathbf{U} \in \mathbb{C}^{N\times N}$ unitary and $\mathbf{V} \in \mathbb{C}^{N\times N}$ unitary and $\mathbf{D} \in \mathbb{C}^{N\times N}$ diagonal such that:

$$\mathbf{A} \;=\; \mathbf{U}\,\mathbf{D}\,\mathbf{V}^H. \tag{10}$$

First, we consider $\mathbf{A}\mathbf{A}^H \in \mathbb{C}^{N\times N}$. This is a symetric matrix. Therefore, there exist $\mathbf{U} \in \mathbb{C}^{N\times N}$ unitary and $\mathbf{D}_2 \in \mathbb{C}^{N\times N}$ diagonal satisfying:

$$\mathbf{A}\mathbf{A}^H \;=\; \mathbf{U}\,\mathbf{D}_2\,\mathbf{U}^H \tag{11}$$

In other words, there exist N vectors $\mathbf{u}_i$ and N scalars λ_i such that:

$$\mathbf{A}\mathbf{A}^H\,\mathbf{u}_i \;=\; \lambda_i\,\mathbf{u}_i, \quad \text{for all } i = 1, ..., N. \tag{12}$$

where the $\mathbf{u}_i$'s are the columns of the matrix $\mathbf{U}$ and λ_i are the coefficients of the diagonal matrix $\mathbf{D}_2$. The $\mathbf{u}_i$ satisfies the orthogonal property:

$$\mathbf{U}^H\mathbf{U} \;=\; \mathbf{I} \quad\Longleftrightarrow\quad \mathbf{u}_i^H\,\mathbf{u}_j \;=\; 1 \text{ if } i = j \quad \text{and} \quad 0 \text{ else}. \tag{13}$$

We define the $\mathbf{v}_i$ in the following way :

$$i \in I = \{i \in \mathbb{N} \text{ with } 1 \leqslant i \leqslant N \text{ and } \mathbf{A}^H\,\mathbf{u}_i \neq 0\}, \quad \mathbf{v}_i \;=:\; \frac{\mathbf{A}^H\,\mathbf{u}_i}{\|\mathbf{A}^H\,\mathbf{u}_i\|}. \tag{14}$$

Remarking that $\forall i \in I$:

$$\begin{cases} \|\mathbf{A}^H\,\mathbf{u}_i\|^2 \;=\; (\mathbf{A}^H\,\mathbf{u}_i; \mathbf{A}^H\,\mathbf{u}_i) \;=\; (\mathbf{A}\mathbf{A}^H\,\mathbf{u}_i;\;\mathbf{u}_i) \;=\; \lambda_i, \\ (\mathbf{A}^H\,\mathbf{u}_i; \mathbf{A}^H\,\mathbf{u}_j) \;=\; (\mathbf{A}^H\,\mathbf{u}_i; \mathbf{A}^H\,\mathbf{u}_j) \;=\; (\mathbf{A}\mathbf{A}^H\,\mathbf{u}_i;\;\mathbf{u}_j) \;=\; \lambda_i\,(\mathbf{u}_i;\;\mathbf{u}_j), \end{cases} \tag{15}$$

leads to for all i and j in I:

$$\mathbf{v}_i^H\,\mathbf{v}_j \;=\; 1 \text{ if } i = j \quad \text{and} \quad 0 \text{ else}. \tag{16}$$

Bitte wenden!

By the Gram-Schmidt procedure, we complete the system of vectors $(\mathbf{v}_i)_{i\in I}$ so that the completed version $(\mathbf{v}_i)_{i=1}^{N}$ keep the orthogonality property (16). By construction, one has:

$$\sqrt{\lambda_i}\,\mathbf{v}_i \;=\; \mathbf{A}^H\,\mathbf{u}_i, \quad \forall i = 1, ..., N. \tag{17}$$

Now, we denote by $\mathbf{V} \in \mathbb{C}^{N\times N}$ and $\mathbf{D} \in \mathbb{C}^{N\times N}$ the matrices given by:

$$\mathbf{V} \;=\; (\mathbf{v}_1|\mathbf{v}_2|...|\mathbf{v}_N) \quad \text{and} \quad \mathbf{D} \;=\; \sqrt{\mathbf{D}_2}. \tag{18}$$

The matrix $\mathbf{V} \in \mathbb{C}^{N\times N}$ is unitary and the matrix $\mathbf{D} \in \mathbb{C}^{N\times N}$ is diagonal with $\mathbf{D}_{i,i} = \sqrt{\lambda_i}$. Finally, equations (16) and (17) lead to:

$$\mathbf{D} \;=\; \mathbf{V}^H\mathbf{A}^H\,\mathbf{U} \quad\Longrightarrow\quad \mathbf{V}\,\mathbf{D}\,\mathbf{U}^H \;=\; \mathbf{A}^H \quad\Longrightarrow\quad \mathbf{A} \;=\; \mathbf{U}\,\mathbf{D}\,\mathbf{V}^H. \tag{19}$$

We say that the $\sqrt{\lambda_i}$, $i = 1, ..., N$, are the singular values of the matrix $\mathbf{A}$.

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Matlab command:

[U,S,V] = svd(A) produces a diagonal matrix S, of the same
  dimension as X and with nonnegative diagonal elements in
  decreasing order, and unitary matrices U and V so that
  A = U*S*V'.
S = svd(A) returns a vector containing the singular values.
Smin = svds(A,1,0) returns the smallest singular value.
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