Serie 10

Theoretical Exercises: 1.

Practical Exercise: 2.

1. (Inverse spectral estimates)

Let $\Lambda = [-1; 1]$, and $p \in \mathbb{N}$.

Prove that the following inverse inequality is optimal with respect to p:

$$|u_N|_{H^1(\Lambda)} \leqslant C\ p^2\ \|u_N\|_{L^2(\Lambda)}, \qquad \forall u_N \in \mathcal{P}_p(\Lambda). \tag{1}$$

2. (Poincaré inequality)

Let Ω be the triangle of $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Due to the Poincaré inequality there exists $C > 0$ such that:

$$\|u\|_{L^2(\Omega)} \leqslant C\ |u|_{H^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{2}$$

Approximate the smallest C satisfying (2).

Hint: We first remark that C is given by:

$$C^2 = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2_{L^2(\Omega)}}{|u|^2_{H^1(\Omega)}}. \tag{3}$$

We introduce the Galerkin space $\mathcal{S}^0_{1,0}(\mathcal{M}) \subset H_0^1(\Omega)$, where $\mathcal{M}$ is a shape regular mesh of Ω. The constant C is approximated by C_N:

$$C_N^2 = \sup_{u_N \in \mathcal{S}^0_{1,0}(\mathcal{M})\setminus\{0\}} \frac{\|u_N\|^2_{L^2(\Omega)}}{|u_N|^2_{H^1(\Omega)}}. \tag{4}$$

To compute C_N, we introduce the Lagrangian:

$$\mathcal{L}(\alpha, \lambda) = \alpha^T\ \mathsf{M}\ \alpha - \lambda\ \alpha^T\ \mathsf{K}\ \alpha \tag{5}$$

Here, we have associated to $u_N \in \mathcal{S}^0_{1,0}(\mathcal{M})$ the vector α

$$u_N \;=\; \sum_{i=1}^{M} \alpha^i \; b_N^i \tag{6}$$

where the $(b_N^i)'s$ denote the standard hat functions.

The matrices M and K are the mass and stiffness matrices.

$$\mathsf{M}_{i,j} \;=\; \int_\Omega b_N^i(x)\; b_N^j(x)\; dx \quad \text{and} \quad \mathsf{K}_{i,j} \;=\; \int_\Omega \nabla b_N^i(x)\; \nabla b_N^j(x)\; dx \tag{7}$$

Finally C_N^2 can be computed as the largest λ satisfying:

$$\mathsf{M}\alpha \;-\; \lambda \mathsf{K}\alpha \;=\; 0 \quad \text{with} \quad \alpha^T \mathsf{K}\alpha = 1. \tag{8}$$

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EIGS  Find a few eigenvalues and eigenvectors of a
      matrix using ARPACK
   EIGS(A) returns a vector of A's 6 largest magnitude
      eigenvalues. A must be square and should be large
      and sparse.

   EIGS(A,B) solves the generalized eigenvalue problem
      A*V == B*V*D. B must be symmetric (or Hermitian)
      positive definite and the same size as A.

   EIGS(A,K) and EIGS(A,B,K) return the K largest
      magnitude eigenvalues.
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Matlab files (that have to be completed) can be found at:

http://www.sam.math.ethz.ch/ tordeux/LFE2/

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoretical exercises and 50% practical exercises (MATLAB) should be solved.