Serie 1

Context: Galerkin approximation.

Theoretical Exercises: 1.–4.

Practical Exercise: none

1. Let $(V, (.,.)_V)$ be a real Hilbert space, $\mathsf{a} : V \times V \to \mathbb{R}$ be a V-elliptic continuous bilinear form, $\mathsf{b} : V \times V \to \mathbb{R}$ be a bilinear continuous form and $f : V \to \mathbb{R}$ be a linear form of V. We consider the problem (P):

$$\text{Find } u_\epsilon \in V \text{ such that } \forall v \in V : \ \mathsf{a}_\epsilon(u,v) \ = \ f(v), \tag{1}$$

with

$$\epsilon \in \mathbb{R} \text{ and } \mathsf{a}_\epsilon(u,v) \ = \ \mathsf{a}(u,v) \ + \ \epsilon\, \mathsf{b}(u,v)_V. \tag{2}$$

We introduce the following notations:

$$\left\{ \begin{array}{llll} \alpha \ = \ \inf\limits_{v\in V\setminus\{0\}} \dfrac{|\mathsf{a}(v,v)|}{\|v\|^2}, & & \|\mathsf{a}\| \ = \ \sup\limits_{v\in V\setminus\{0\}} \sup\limits_{w\in V\setminus\{0\}} \dfrac{|\mathsf{a}(v,w)|}{\|v\|\|w\|}, \\ \|\mathsf{b}\| \ = \ \sup\limits_{v\in V\setminus\{0\}} \sup\limits_{w\in V\setminus\{0\}} \dfrac{|\mathsf{b}(v,w)|}{\|v\|\|w\|}, & & \|f\| \ = \ \sup\limits_{v\in V\setminus\{0\}} \dfrac{|f(v)|}{\|v\|}. \end{array} \right. \tag{3}$$

(i) Prove that for $\epsilon_0 = \alpha / \|\mathsf{b}\|$

$$\forall \epsilon \in \mathbb{R} \text{ with } |\epsilon| < \epsilon_0 \text{ the problem (P) has a unique solution.} \tag{4}$$

(ii) For $|\epsilon| < \epsilon_0$, prove the following error estimate:

$$\|u_\epsilon - u_0\| \ \leq \ \frac{|\epsilon|}{\epsilon_0 - |\epsilon|} \frac{\|f\|}{\alpha} \tag{5}$$

(iii) What is the limit in V of u_ϵ when $\epsilon \to 0$?

Bitte wenden!

2. Let $(U, (.,.)_U)$, $(V, (.,.)_V)$ and $(W, (.,.)_W)$ be real Hilbert spaces with $U \subset V \subset W$ and

$$\forall v \in V,\ \|v\|_W \ \leq\ C_1\, \|v\|_V \quad \text{and} \quad \forall v \in U,\ \|v\|_V \ \leq\ C_2\, \|v\|_U \tag{6}$$

Let $(V_n)_{n\in\mathbb{N}^*}$ be a family of finite dimensional subspaces of V satisfying:

$$\begin{aligned} \forall v \in V, &\quad \inf_{v_n \in V_n} \|v_n - v\|_V \ \leq \|v\|_V, \\ \forall v \in U, &\quad \inf_{v_n \in V_n} \|v_n - v\|_V \ \leq \frac{C_3}{n}\|v\|_U. \end{aligned} \tag{7}$$

Let a be a continuous bilinear form of V satisying the inf-sup condition

$$\begin{aligned} \exists \alpha > 0 \ / &\qquad \inf_{v\in V\setminus\{0\}} \sup_{w \in V\setminus\{0\}} \frac{|\mathsf{a}(v,w)|}{\|v\|_V\|w\|_V} \geq \alpha, \\ \forall w \in V \setminus \{0\}, &\qquad \sup_{v \in V\setminus\{0\}} \frac{|\mathsf{a}(v,w)|}{\|v\|_V\|w\|_V} > 0, \end{aligned} \tag{8}$$

and the discreete inf-sup condition

$$\begin{aligned} \exists \alpha > 0 \ / \ \forall n \in \mathbb{N}^*, &\qquad \inf_{v_n\in V_n\setminus\{0\}} \sup_{w_n \in V_n\setminus\{0\}} \frac{|\mathsf{a}(v_n,w_n)|}{\|v_n\|_V\|w_n\|_V} \geq \alpha, \\ \forall n \in \mathbb{N}^*,\ \forall w_n \in V_n \setminus \{0\}, &\qquad \sup_{v_n \in V_n\setminus\{0\}} \frac{|\mathsf{a}(v_n,w_n)|}{\|v_n\|_V\|w_n\|_V} > 0, \end{aligned} \tag{9}$$

Let $f : W \to \mathbb{R}$ be a linear form satisfying:

$$\sup_{w\in W\setminus\{0\}} \frac{|f(w)|}{\|w\|_W} \ = \ \|f\|_{W'} \tag{10}$$

We consider the problems:

$$\begin{cases} \text{Problem } (P): \ \text{find } u \in V \text{ such that } \forall v \in V: \ \mathsf{a}(u,v) \ = \ f(v), \\ \text{Problem } (P_n): \ \text{find } u_n \in V \text{ such that } \forall v_n \in V: \ \mathsf{a}(u_n,v_n) \ = \ f(v_n). \end{cases} \tag{11}$$

(i) Prove that the problems (P) and (P_n)'s have unique solutions and the following stability results:

$$\|u\|_V \ \leq \ \frac{C_1}{\alpha}\|f\|_{W'} \text{ and } \ \|u_n\|_V \ \leq \ \frac{C_1}{\alpha}\|f\|_{W'} \tag{12}$$

(ii) Suppose that $u \in U$ and prove the following error estimate:

$$\|u - u_n\|_V \ \leq \ \left(1 + \frac{\|\mathsf{a}\|}{\alpha}\right) \frac{C_3}{n}\|u\|_U \tag{13}$$

Siehe nächstes Blatt!

3. Let $(V, (.,.)_V)$ be an infinite dimensional Hilbert space. Prove that there exists no linear space V_n satisfying:

- V_n is a finite dimensional subspace of V.
- For all $v \in V$, $\inf_{v_n \in V_n} \|v - v_n\|_V \ \leq \ \alpha \, \|v\|_V$ with $\alpha < 1$.

Hint: Take a v in the orthogonal of V_n in V.

4. Let $(V, (.,.)_V)$ be a real Hilbert space, $\mathsf{a} : V \times V \to \mathbb{R}$ be a V-elliptic continuous bilinear form and $f : V \to \mathbb{R}$ be a linear continuous form. We introduce the following notations:

$$\begin{aligned} \|\mathsf{a}\| \ &= \ \sup_{u \in V \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{\mathsf{a}(u,v)}{\|u\|_V \|v\|_V}, \quad \alpha \ = \ \inf_{u \in V \setminus \{0\}} \frac{\mathsf{a}(u,u)}{\|u\|_V^2} \\ \|f\| \ &= \ \sup_{v \in V \setminus \{0\}} \frac{f(u)}{\|u\|_V} \end{aligned} \tag{14}$$

We consider the problem (P)

$$\left\{ \begin{array}{l} \text{Find } u_1, \ u_2 \in V \text{ such that } \forall v_1, \ v_2 \in V : \\ \ \mathsf{a}(u_1, v_1) \ + \ \mathsf{a}(u_2, v_1) \ + \ \mathsf{a}(u_2, v_2) \ = f(v_2). \end{array} \right. \tag{15}$$

Prove that the problem (P) has a unique solution which satisfies:

$$\|u_2\|_V \ \leq \ \frac{\|f\|}{\alpha} \text{ and } \|u_1\|_V \ \leq \ \frac{\|\mathsf{a}\| \ \|f\|}{\alpha^2} \tag{16}$$

Tutorial: Thursday 10–11 HG E5, **Starting time:** Thursday, 3.10

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Testat requirement: 50% of the theoritical exercises and 50% practical exercises (MATLAB) should be solved.