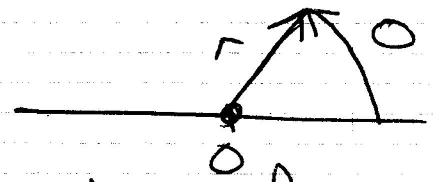
Serie 9:
Exercise 1:

$\Delta u = 0 \quad x \in \mathbb{R},\ y > 0 \iff \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} u + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u = 0$ (a)

$u(x,0) = 0 \quad x > 0 \iff u(r,0) = 0 \quad \forall r > 0$ (b)

$\frac{\partial u}{\partial y}(x,0) = 0 \quad x < 0 \iff \frac{\partial u}{\partial \theta}(r,\pi) = 0 \quad \forall r > 0$ (c)



Separation of variables

$u_n(r,\theta) = f_n(r)\, g_n(\theta)$

$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\left(f_n(r)\right) \cdot g_n(\theta) + \frac{1}{r^2} f_n(r)\, g_n''(\theta) = 0$ (a)

$g_n(0) = 0$ (b) and $g_n'(\pi) = 0$ (c)

$$\frac{r \frac{d}{dr}\left[r \frac{d}{dr} f_n(r)\right]}{f_n(r)} + \frac{g_n''(\theta)}{g_n(\theta)} = 0$$

(does not depend on θ) (does not depend on r)

$\longrightarrow g_n''(\theta) + (\lambda_n)\, g_n(\theta) = 0$

f $\lambda_n < 0 \qquad g_n(\theta) = A_n \cosh(\sqrt{-\lambda_n}\,\theta) + B_n \sinh(\sqrt{\lambda_n}$

$g_n(0) = 0$ (b) $\Rightarrow A_n = 0$

$g_n'(\pi) = 0$ (c) $\Rightarrow B_n = 0$

$\Big\}$ impossible

if $\lambda_n = 0 \Longrightarrow$ impossible also!

$$g_n(\theta) = A_n + B_n t \text{ with } A_n = B_n = 0$$

if $\lambda_n > 0$

$$g_n(\theta) = A_n \cos(\sqrt{\lambda_n}\,\theta) + B_n \sin(\sqrt{\lambda_n}\,\theta)$$

This leads to:

$$g_n(0) = 0 \quad (b) \Rightarrow A_n = 0$$
$$g_n'(\pi) = 0 \quad (c) \Longrightarrow \sqrt{\lambda_n} = \left(\tfrac{1}{2} + n\right) \text{ with } n \in \mathbb{N}_0$$

Hence $$g_n(\theta) = B_n \sin\sqrt{(n+\tfrac{1}{2})}\,\theta$$

A_n $$r\frac{d}{dr}\, r\frac{d}{dr} f_n(r) + \lambda_n f_n(r) = 0$$

$$\hookrightarrow r\frac{d}{dr}\, r\frac{d}{dr} f_n(r) + (n+\tfrac{1}{2})^2 f_n(r) = 0$$

$$f_n(r) = C_n r^{n+\frac{1}{2}} + D_n r^{-n+\frac{1}{2}}$$

The singular functions are $(n \in \mathbb{N}_0)$

$$\boxed{u_n(r,\theta) = r^{n+\frac{1}{2}} \sin\left[(n+\tfrac{1}{2})\,\theta\right] \in H^1_{bc}(\Omega)}$$

$$\sigma_n(r,\theta) = r^{-(n+\frac{1}{2})} \sin\left[(n+\tfrac{1}{2})\,\theta\right] \notin H^1_{bc}(\Omega)$$

Exercise 2:

(i) We consider u in $H^1(\Omega)$

$$\| u(\varphi_N^i(t))\|_{L^2(0,1)} = \| u\|_{L^2(a_N^i, a_N^{\sigma(i)})} \cdot C_i$$

Due to the trace theorem:

$$\| u(\varphi_N^i(t))\|_{L^2(0,1)} \leqslant C_i \| u\|_{H^1(\Omega)}$$

The triangular inequality leds to:

$$\|Tu\|_{H^1(\Omega)} \leqslant \sum_{i=1}^{N} \left| \int_0^1 (4-6t)\, u(\varphi_N^i(t))\, dt \right| \, \|b_N^i\|_{H^1(\Omega)}$$

$$\underset{\text{Cauchy-Schwartz}}{\leqslant} \sum_{i=1}^{N} C_i \|4-6t\|_{L^2(0,1)} \cdot \|u\|_{H^1(\Omega)} \cdot \|b_N^i\|_{H^1(\Omega)}$$

$$\leqslant \left(\sum_{i=1}^{N} C_i \|4-6t\|_{L^2(0,1)} \|b_N^i\|_{H^1(\Omega)} \right) \|u\|_{H^1(\Omega)}$$

$$\leqslant C(h) \|u\|_{H^1(\Omega)}.$$

$\hookrightarrow$ T is continuous in $H^1(\Omega)$

T is not continuous in $L^2(\Omega)$:
The trace of u is not defined on $[a_N^i ; a_N^{\sigma(i)}]$. Hence, the operator is not defined in $L^2(\Omega)$.

(ii) The key point of the proof is the following

$$T b_N^i = b_N^i$$

Indeed

(α) $b_N^i(\varphi_N^i(t)) = (1-t)$

$$\Longrightarrow \int_0^1 (4-6t)\, b_N^i(\varphi_N^i(t))\,dt = \int_0^1 (4-6t)(1-t)\,dt$$

$$= \int_0^1 4 - 10t + 6t^2\,dt$$

$$= 4 - 5 + 2 = 1$$

(β) $b_N^i(\varphi_N^j(t)) = 0$ if $i \neq \sigma(j)$

$b_N^i(\varphi_N^j(t)) = t$ if $i = \sigma(j)$

if $i \neq \sigma(j)$

$$\int_0^1 (4-6t)\, b_N^i(\varphi_N^j(t))\,dt = 0$$

if $i = \sigma(j)$

$$\int_0^1 (4-6t)\, b_N^i \varphi_N^j(t)\,dt = \int_0^1 (4-6t)\, t\,dt$$

$$= 0$$

This implies: $Tu = \sum_{j=1}^{N} \int_0^1 (4-6t)\, b_N^i(\varphi_N^j(t))\,dt\, b_N^j$

$$\boxed{T b_N^i = b_N^i}$$

$= 0$ if $i \neq j$

$= 1$ if $i = j$

This leads to:

$$T^2 u = T\left[\sum_{i=1}^{N} \int_0^1 (4-6t)\, u(\gamma_N^i(t))\, dt \;\; b_N^i\right]$$

$$= \left[\sum_{i=1}^{N} \int_0^1 (4-6t)\, u(\gamma_N^i(t))\, dt \;\; T b_N^i\right]$$

$$= \left[\sum_{i=1}^{N} \int_0^1 (4-6t)\, u(\gamma_N^i(t))\, dt \;\; b_N^i\right]$$

$$= Tu. \qquad \forall u \in H^1(\Omega)$$

(iii) Let i be the index of a node on the boundary as $[a_N^i ; a_N^{\alpha(i)}] \in \partial\Omega \Rightarrow u(\gamma_N^i(t))=$

$$\Rightarrow Tu = \sum_{i / a_N^i \notin \partial\Omega} \int_0^1 (4-6t)\, u(\gamma_N^i(t))\, dt\, b_N^i$$

$$\Longrightarrow Tu = 0 \text{ on } \partial\Omega .$$

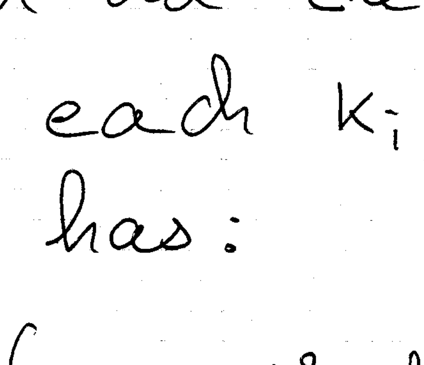
Exercise 3: (Inverse inequality)

- On $\hat{K}$, the unit triangle, for any edge $\hat{e}$ $u_n \mapsto \left(\int_{\hat{e}} |u_n|^2 \, dx\right)^{\frac{1}{2}}$ is a semi-norm of $S_1^0(\mathcal{M})$. Hence as $S_0^1(\mathcal{M})$ is finite dimensional

$$\left(\int_{\hat{e}} |u_n|^2 \, dx\right)^{1/2} \leq C_1 \left(\int (u_n)^p \, dx\right)^{1/p}$$

- Due to a scaling argument and via the shape regularity, for each K_i touching the boundary one has:

$$\left(\int_{e_i} |u_n|^2 \, dx\right)^{1/2} \frac{1}{h^{1/2}} \leq C_1 \frac{1}{h^{2/p}} \left(\int_{K_i} (u_n)^p \, dx\right)^{1/p}$$



Hence

$$\left(\int_{e_i} |u_N|^2 dx\right) \leqslant C_1\left(h^{\frac{1}{2}-\frac{2}{p}}\right)^2 \left(\left(\int_{K_i} |u_N|^p dx\right)^{\frac{2}{p}}\right)$$

We sum over the K_i (touching the boundary

$$\sum_{K_i} \int |u_N|^2 dx \leqslant C\left(h^{\frac{1}{2}-\frac{2}{p}}\right)^2 \sum_{K_i} \left(\int_{K_i} |u_N|^p dx\right)^{2/p}$$

Due to the Jensen inequality, one has:

$$\|u_N\|^2_{L^2(\partial\Omega)} \leqslant C\left(h^{\frac{1}{2}-\frac{2}{p}}\right)^2 \left(\sum_{K_i} \int_{K_i} (u_N)^p\right)^{\frac{2}{p}}$$

$$\left(\sum\left(x^{\frac{2}{p}}\right) \leqslant \left(\sum x\right)^{\frac{2}{p}} \right)$$

$$\leqslant C\left(h^{\frac{1}{2}-\frac{2}{p}}\right)^2 \left(\sum_{K_i \in \Omega} \int_{K_i} (u_N)^p\right)^{2/p}$$

This implies:

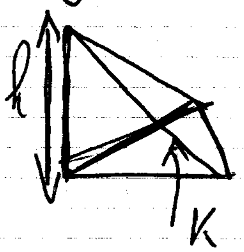
$$\|u_N\|_{L^2(\partial\Omega)} \leqslant C \; h^{\frac{1}{2}-\frac{2}{p}} \; \|u_N\|_{L^2(\Omega)}.$$

Exercise 4:

We first start with a remark:

$$\int_K p_N \, dx = \int_{\substack{K \\ num}} p_N \, dx \qquad \text{for all } p_N \in P_1(K)$$

Indeed the volume of a pyramid is given by



$$\frac{|K|}{3} h$$

Hence by linear combination, we have:

$\forall p_n \in P_1(K)$ / K triangle with vertices a^1, a^2, a^3

$$\int p_n = \frac{|K|}{3}\left(p_N(a^1) + p_N(a^2) + p_N(a^3)\right)$$

The first direct consequeen of this remark is

$$\int_K Du_N \, Dv_N = \int_{\substack{K \\ num}} Du_N \, Dv_N \qquad \forall u_N, v_N \in P_1(K)$$

as $(Du_N \, Dv_N) \in P_0(K)$

This implies that:

$$a(u_N, v_N) - a_N(u_N, v_N) = \int_K u_N v_N - \int_{\substack{K \\ num}} u_N v_N .$$

We work now on the unit triangle

Let $\hat{u}_n, \hat{v}_n$ be two elements of $P_1(\hat{K})$

$$\int_{\hat{K}} \hat{u}_n \hat{v}_n - \int_{\hat{K}}^{num} \hat{u}_n \cdot \hat{v}_N = \int_{\hat{K}} (\hat{u}_N - \hat{p})(\hat{v}_N - \hat{q})\, dx - \int_K^{num} \cdots$$

$$+ \underbrace{\int_K (\hat{p}\hat{v}_N + \underbrace{\hat{u}_N \hat{q} - \hat{p}\hat{q}}_{\in P_1(\hat{K})})\, dx - \int_{\hat{K}}^{num} \cdots}_{\text{exact integration } = 0.}$$

$$\left| \int_{\hat{K}} \hat{u}_N \hat{v}_N - \int_{\hat{K}}^{num} \hat{u}_N \hat{v}_N \right| \leq C \| \hat{u}_N - \hat{P} \|_{L^2(\hat{K})} \| \hat{v}_N - \hat{q} \|_{L^2(\hat{K})} \cdot$$

$$\leq C_1 |\hat{u}_N|_{H^1(\hat{K})} \cdot |\hat{v}_N|_{H^1(\hat{K})}$$

Via the scaling argument and due to the shape regularity; we have

$$\left| \int_{K_i} u_N v_N - \int_{K_i}^{num} u_N v_N \right| \leq C h^2 |u_N|_{H^1(K_i)} |v_N|_{H^1(K_i)} \quad \forall K_i \in \mathcal{M}$$

Hence, one has:

$$|a(u_N, v_N) - a_N(u_N, v_N)| \leq C h^2 \sum_{K_i \in \mathcal{M}} |u_N|_{H^1(K_i)} |v_N|_{H^1(K_i)}$$

Due to the Cauchy-schwartz inequality, we have:

$$|a(u_N, v_N) - a_N(u_N, v_N)| \leq C h^2 \left(\sum_{K_i \in \mathcal{M}} |u_N|^2_{H^1(K_i)} \right)^{1/2} \left(\sum_{K_i \in \mathcal{M}} |v_N|^2_{H^1(K_i)} \right)^{1/2}$$

$$\leq C h^2 |u_N|_{H^1(\Omega)} |v_N|_{H^1(\Omega)}$$