Série 8:

Exercise 1:

$$\int_{\phi_h(\hat{K})} \left| \frac{\partial^m u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \right|^p dx_1 \cdots dx_2 = \int_{\phi_h(\hat{K})} \left| \frac{1}{h^m} \frac{\partial^m \hat{u}}{\partial \hat{x}_1^{\alpha_1} \cdots \partial \hat{x}_d^{\alpha_d}} \right|^p h^d d\hat{x}$$

with $\sum \alpha_i = m$

$$\left\| \frac{\partial^m u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \right\|^p_{L^p(\phi_h(\hat{K}))} = h^{d-mp} \left\| \frac{\partial^m \hat{u}}{\partial \hat{x}_1^{\alpha_1} \cdots \partial \hat{x}_d^{\alpha_d}} \right\|^p_{L^p(\hat{K})}$$

By summation, one has:

$$|u|^p_{W^{m,p}(\phi_h(\hat{K}))} = h^{d-mp} |\hat{u}|^p_{W^{m,p}(\hat{K})}$$

This unplies:

$$|u|_{W^{m,p}(\phi_h(\hat{K}))} = h^{\frac{d-mp}{p}} |\hat{u}|_{W^{m,p}(\hat{K})}$$

Exercise 2:

We use that for all $K \in \mathcal{M}$

$$H^m(K) \overset{c}{\hookrightarrow} H^k(K) \qquad (*)$$

Let us consider a bounded sequence of $H^m(\mathcal{M})$.

As there is ~~associated~~ a finite number of triangles, we can associate to the mesh the numbering

$$\mathcal{M} = \{K_i\}_{i=1}^{N}.$$

We act by induction:

~~The~~ We consider a bounded sequence (u_i) of $H^m(\mathcal{M})$. Therefore this sequence is also bounded in $H^m(K_i)$. Due to $(*)$, we can extract from (u_i) the sequence $(u_i^1)_{i=1}^{\infty}$ wich converges in $H^k(K_1)$

We suppose that there exist a sequence $(u_i^j)_{i=1}^{\infty}$ extracted from $(u_i)_{i=1}^{\infty}$ such that

$(u_i^j)_{i=1}^{\infty}$ converges $(i \to \infty)$ in $H^k(K_j)$

for $j = 1, \dots, d$

As $(u_i^j)_{i=1}^{\infty}$ is bounded in $H^m(K_{j+1})$,
we can extract $(u_i^{j+1})_{i=1}^{\infty}$ such that
$$(u_i^{j+1})_{i=1}^{+\infty} \text{ converges in } H^k(K_{j+1}).$$
Moreover as (u_i^{j+1}) as been extracted from (u_i^j), it converges also in $H^k(K_{j'})$ for $j' \leq j$.

Acting this way, we extract $(u_i^N)_{i=1}^{\infty}$ such that
$$(u_i^N) \text{ converges in } H^k(\mathcal{M}).$$

Exercise 3: We first recall some properties of P_i

$$\forall v \in H^1(\Omega), \quad \|P_i v\|_{H^1(\Omega)} \leqslant C_1 \|v\|_{H^1(\Omega)}$$

$$\forall v \in H^2(\Omega), \quad \|P_i v - v\|_{H^1(\Omega)} \leqslant C_2 \|v\|_{H^2(\Omega)} 2^{-i}$$

Let us fix $u \in H^1(\Omega)$

As $H^2(\Omega)$ is dense in $H^1(\Omega)$, one has

$$\exists\, \omega \in H^2(\Omega) \;/\; \|u - \omega\| \leqslant \left(\frac{\varepsilon}{1+C_1}\right) \cdot \frac{1}{2}$$

(ω is fixed)

$$\begin{aligned}\|P_i u - u\|_{H^1(\Omega)} &\leqslant \|P_i u - P_i \omega\|_{H^1(\Omega)} + \|u - \omega\|_{H^1(\Omega)} \\ &\quad + \|P_i \omega - \omega\|_{H^1} \\ &\leqslant (C_1 + 1)\, \|u - \omega\|_{H^1(\Omega)} \\ &\quad + \|\omega\|_{H^2(\Omega)} \cdot C_2\, 2^{-i}\end{aligned}$$

For all $i \in \mathbb{N}_0 \;/\; i > i_0 \left(:= \ln\left(\dfrac{2C_2}{\|\omega\|_{H^2(\Omega)}}\right) / \ln(2) \right)$

$$\|P_i u - u\|_{H^1(\Omega)} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

$$\forall \varepsilon > 0 \;\; \exists\, i_0 \in \mathbb{R} \;/\; \forall i > i_0 \quad \|P_i u - u\| \leqslant \varepsilon$$

Hence,

$$P_i u \longrightarrow u \quad \text{strongly in } H^1(\Omega)$$

The same technique can be applied to Q_i

Exercise 4:

Let $\mathcal{M}$ be the mesh: $\mathcal{M} = \{K_j\}_{j=1}^{n}$

Let ϕ_j be the linear mapping $\hat{K} \longrightarrow K_j$

$$\left\| \sum_{i=1}^{N} \alpha_i b^i \right\|^2_{L^2(\Omega)} = \sum_{j=1}^{M} \left\| \sum_{i=1}^{N} \alpha_i b^i \right\|^2_{L^2(K_j)}$$

$$= \sum_{j=1}^{M} \left\| \sum_{i=1}^{3} \alpha^i_{K_j} b^i_{K_j} \right\|^2_{L^2(K_j)}$$

$$= \sum_{j=1}^{M} |D\phi_j| \left\| \sum_{i=1}^{3} \alpha^i_{K_j} \hat{b}^i_{\hat{K}} \right\|^2_{L^2(\hat{K})}$$

(equivalence of norms $\underline{C} \ldots \leq \ldots \leq \bar{C} \ldots$)
$$\simeq \sum_{j=1}^{M} |D\phi_j| \left(\sum_{i=1}^{3} |\alpha^i_{K_j}|^2 \, \|\hat{b}^i_{\hat{K}}\|^2_{L^2(\hat{K})} \right)$$

$$\simeq \sum_{j=1}^{M} \left(\sum_{i=1}^{3} (\alpha^i_{K_j})^2 \, \|b^i_{K_j}\|^2_{L^2(K_j)} \right)$$

$$\simeq \sum_{j=1}^{M} \left(\sum_{i=1}^{N} (\alpha_i)^2 \, \|b_i\|^2_{L^2(K_j)} \right)$$

$$\simeq \sum_{i=1}^{N} (\alpha_i)^2 \sum_{j=1}^{M} \|b_i\|^2_{L^2(K_j)}$$

$$\simeq \sum_{i=1}^{N} (\alpha_i)^2 \|b_i\|^2_{L^2(\Omega)}$$

Exercise 5:

This is a direct consequence of the interpo-
-lation theorem (take $T = Id - Q_H$)

Theorem: Let T be a continuous linear operator

$$T: \; H^m(\Omega) \longrightarrow L^2(\Omega) \qquad (m > k)$$
$$H^k(\Omega) \longrightarrow L^2(\Omega)$$

with $\|Tu\|_{L^2(\Omega)} \leqslant C_1 \|u\|_{H^m(\Omega)}$

$\|Tu\|_{L^2(\Omega)} \leqslant C_2 \|u\|_{H^k(\Omega)}$

then for $s \in [m, k]$, one has:

$$\|Tu\|_{L^2(\Omega)} \leqslant (C_1)^{\frac{k-s}{k-m}} (C_2)^{\frac{s-m}{k-m}} \|u\|_{H^s(\Omega)}.$$

Exercise 6:

We recall the property of I_p (see the slides)

$$\|I_p u - u\|_{H^1(\Lambda)} \leqslant C_k p^{-k} \|u\|_{H^{k+1}(\Lambda)} \quad \forall k \geqslant 0$$

$$\|I_p u - u\|_{L^2(\Lambda)} \leqslant C p^{-k} \|u\|_{H^k(\Lambda)} \quad \forall k \geqslant 0$$

and the commutation properties:

$$\begin{cases} \frac{\partial}{\partial x} I_p^y = I_p^y \frac{\partial}{\partial x} ; \\ \frac{\partial}{\partial y} I_p^x = I_p^x \frac{\partial}{\partial y} \end{cases}$$

$$|u-\Pi_p u|^2_{H^1(\hat{K})} = \underbrace{\left\|\frac{\partial}{\partial x}(u-\Pi_p u)\right\|^2_{L^2(\hat{K})}}_{i} + \underbrace{\left\|\frac{\partial}{\partial y}(u-\Pi_p u)\right\|^2_{L^2(\hat{K})}}_{ii}$$

(i) $$\left\|\frac{\partial}{\partial x}(u-\Pi_p u)\right\|_{L^2(\hat{K})} \leqslant \underbrace{\left\|\frac{\partial}{\partial x}(u-I_p^x u)\right\|_{L^2(\hat{K})}}_{a} + \underbrace{\left\|\frac{\partial}{\partial x}(I_p^x u - I_p^x I_p^y u)\right\|_{L^2(\hat{K})}}_{b}$$

(i,a) $$\left\|\frac{\partial}{\partial x}(u-I_p^x u)\right\|^2_{L^2(\hat{K})} = \left\| \left\| \left(u(\cdot,y)-I_p u(\cdot,y)\right)' \right\|^2_{L^2_x(\Lambda)} \right\|^2_{L^2_y(\Lambda)}$$

$$\leqslant \left\| C_1 p^{-k+1} \|u\|^2_{H^k_x(\Lambda)} \right\|^2_{L^2_y(\Lambda)} \qquad \forall k \geqslant 1$$

$$\leqslant \left(C p^{-k+1}\right)^2 \left\| \|u\|^2_{H^k_x(\Lambda)} \right\|^2_{L^2_y(\Lambda)}$$

$$\left\|\frac{\partial}{\partial x}(u-I_p^x u)\right\|_{L^2(\hat{K})} \leqslant C_1 p^{-k+1} \|u\|^2_{H^k(\hat{K})}$$

(i,b) $$\left\|\frac{\partial}{\partial x}(I_p^x u - I_p^x I_p^y u)\right\|^2_{L^2(\Omega)} = \left\| \left\| \left(I_p\left(u(\cdot,y) - I_p^y u(\cdot,y)\right)\right)' \right\|^2_{L^2_x(\Lambda)} \right\|^2_{L^2_y(\Lambda)}$$

$$\leqslant C \left\| \|u(\cdot,y) - I_p^y u(\cdot,y)\|^2_{H^1_x(\Lambda)} \right\|^2_{L^2_y(\Lambda)}$$

$$\leqslant C_1 \left\| \|u(x,\cdot) - I_p u(x,\cdot)\|^2_{L^2_y(\Lambda)} \right\|^2_{H^1_x(\Lambda)}$$

$$\leqslant C_1 \left\| C_1 p^{-k+1} \|u\|^2_{H^{k-1}_y(\Lambda)} \right\|^2_{H^1_x(\Lambda)}$$

$$\leqslant \left(C p^{-k+1}\right)^2 \left\| \|u\|^2_{H^{k-1}_y(\Lambda)} \right\|^2_{H^1_x(\Lambda)}$$

$$\left\|\frac{\partial}{\partial x}(I_p^x u - I_p^x I_p^y u)\right\|_{L^2(\Omega)} \leqslant C p^{-k+1} \|u\|_{H^k(\hat{K})}$$

This implies: $\left\|\frac{\partial}{\partial x}(u - \Pi_x u)\right\|_{L^2(\hat{K})} \leqslant C_1 p^{-k+1} \|u\|_{H^k(\hat{K})}$.

One can obtain (ii) using the same technique