Serie 7:

Exercise 1:

a) $L_0 = \frac{1}{0!\,2^0}\left[(x^2-1)^0\right]^{(0)} = 1$

$$L_1(x) = \frac{1}{1!\,2^1}\left[x^2-1\right]^{(1)} = x$$

$$L_2(x) = \frac{1}{2!\,2^2}\left[x^4-2x^2+1\right]^{(2)} = \frac{1}{8}\left(12x^2-4\right) = \frac{3}{2}x^2 - \frac{1}{2}.$$

b) $L'_{n+1}(x) = \frac{1}{(n+1)!\,2^{n+1}}\left[(x^2-1)^{n+1}\right]^{(n+2)}$

$$L'_{n-1}(x) = \frac{1}{(n-1)!\,2^{n-1}}\left[(x^2-1)^{n-1}\right]^{(n)}$$

$$L'_{n+1} - L'_{n-1} = \frac{1}{(n+1)!\,2^{n+1}}\left[\left[(x^2-1)^{n+1}\right]^{(2)} - (n+1)\,n\,4\,(x^2-1)^{n-1}\right]^{(n)}$$

$$= \frac{1}{(n+1)!\,2^{n+1}}\left[(n+1)n4x^2(x^2-1)^{n-1} + 2(n+1)(x^2-1)^n - 4(n+1)\,n\,4\,(x^2-1)^{n-1}\right]^{(n)}$$

$$= \frac{1}{(n+1)!\,2^{n+1}}\left[(n+1)\,n\,4\,(x^2-1)^n + 2(n+1)(x^2-1)^n\right]^{(n)}$$

$$= (2n+1) \times \frac{1}{n!\,2^n}\left[(x^2-1)^n\right]^{(n)} = (2n+1)\,L_n(x)$$

$$\boxed{L_n(x) = \frac{L'_{n+1} - L'_{n-1}}{2n+1}(x)}$$

c) $\forall n \geqslant 1$

$$\int_{-1}^{1} (x^2-1)^n dx = -\int_{-1}^{1} n 2x (x^2-1)^{n-1} x \, dx$$

$$+ \left[x (x^2-1)^n \right]_{-1}^{1}$$

$$= -2n \int_{-1}^{1} x^2 (x^2-1)^{n-1} dx$$

$$= -2n \int_{-1}^{1} (x^2-1)^n - 2n \int_{-1}^{1} (x^2-1)^{n-1}$$

$$\int_{-1}^{1} (x^2-1)^n = \frac{-2n}{2n+1} \int_{-1}^{1} (x^2-1)^{n-1}$$

$$= \frac{(-2n)(-2n+2)\cdots(-2)}{(2n+1)(2n-1)\cdots 3(1)} \int_{-1}^{1} 1 \, dx$$

$$= (-1)^n \frac{2^n n! \, 2^n n!}{(2n+1)(2n)\cdots 2 \cdot 1} \, 2$$

$$= (-1)^n \frac{(2^n n!)^2}{(2n)!} \frac{2}{2n+1}.$$

$$\int_{-1}^{1} L_n^2(z)\, dz = \left(\frac{1}{n!\,2^n}\right)^2 \int_{-1}^{1} \left[(z^2-1)^n\right]^{(n)} \cdot \left[(z^2-1)^n\right]^{(n)} dz$$

$$= \left(\frac{1}{n!\,2^n}\right)^2 (-1)^n \int_{-1}^{1} (z^2-1)^n \left[(z^2-1)^n\right]^{(2n)} dz$$

$$= \left(\frac{1}{n!\,2^n}\right)^2 (-1)^n \int_{-1}^{1} (z^2-1)^n \left[z^{2n}\right]^{(2n)}$$

$$= \left(\frac{1}{n!\,2^n}\right)^2 (-1)^n \int_{-1}^{1} (z^2-1)^n (2n)!\, dz$$

$$= \frac{(2n)!}{(n!\,2^n)^2} (-1)^n \int_{-1}^{1} (z^2-1)^n\, dz$$

$$= \frac{2}{2n+1}.$$

For $n \neq m$, it is enough to deal with $n > m$

$$\int_{-1}^{1} L_m(z) L_n(z) = \left(\frac{1}{m!\,2^m}\right)\left(\frac{1}{n!\,2^n}\right) \int_{-1}^{1} \left[(z^2-1)^n\right]^{(n)} \left[(z^2-1)^m\right]^{(m)}$$

$$= \left(\frac{1}{m!\,2^m}\right)\left(\frac{1}{n!\,2^n}\right) \int_{-1}^{1} (z^2-1)^n \underbrace{\left[(z^2-1)^m\right]^{(m+n)}}_{=0} dz.$$

d) Binomial formula

$$[x f]^{(n)} = x f^{(n)} + n f^{(n-1)}$$

$$x L_n(x) = x\left[(x^2-1)^n\right]^{(n)} \frac{1}{n!\,2^n} = \left[x(x^2-1)^n\right]^{(n)} \frac{1}{n!\,2^n} - n\left[(x^2-1)^n\right]^{(n-1)} \frac{1}{n!\,2^n}$$

$$= \left[2(n+1)\,x\,(x^2-1)^n\right]^{(n)} \cdot \frac{1}{(n+1)!\,2^{n+1}} - \frac{1}{(n-1)!\,2^n}\left[(x^2-1)^n\right]^{(n-1)}$$

$$= \left[(x^2-1)^{n+1}\right]^{(n+1)} \frac{1}{(n+1)2^{n+1}} - \frac{1}{(n-1)!\,2^n}\left[(x^2-1)^n\right]^{(n-1)}$$

$$= L_{n+1}(x) - \frac{1}{(n-1)!\,2^n}\left[(x^2-1)^n\right]^{(n-1)}$$

$$(n+1)!\,2^{n+1}\left(L_{n+1}(x) - L_{n-1}(x)\right) = \left[(x^2-1)^{n+1}\right]^{(n+1)} - 4(n+1)n\left[(x^2-1)^{n-1}\right]^{(n-}$$

$$= \left[\left(\left[x^2-1\right]^{n+1}\right)^{(2)} - 4(n+1)\cdot n\,(x^2-1)^{n-1}\right]^{(n-1)}$$

$$= \left[4(n+1)n\,(x^2-1)^{n-1}\,x^2 + 2(n+1)\left[x^2-1\right]^n - 4(n+1)n\,(x^2-1)^{n-1}\right]^{(n-}$$

$$= \left[4(n+1)\,n + 2(n+1)\right]\left[(x^2-1)^n\right]^{(n-1)}$$

$$= \left[4n^2 + 6n + 2\right]\left[(x^2-1)^n\right]^{(n-1)}$$

$$= (2n+1)(n+1)\,2\left[(x^2-1)^n\right]^{(n-1)}$$

This implies: $L_{n+1}(x) - L_{n-1}(x) = \dfrac{2n+1}{n!\,2^n}\left[(x^2-1)^n\right]^{(n-1)}$.

$$x L_{n+1}(x) = L_{n+1}(x) - \frac{n}{2n+1}\left(L_{n+1}(x) - L_{n-1}(x)\right)$$

$$(2n+1)\, x L_{n+1}(x) = (n+1)\, L_{n+1}(x) - n\, L_{n-1}(x).$$

Exercise 2:

a) By definition

$$\hat{b}_1(x) = \frac{L_0(x) - L_1(x)}{2} \quad \text{and} \quad \hat{b}_2(x) = \frac{L_0(x) + L_1(x)}{2}$$

for $n \geq 2$, one has:

$$\hat{b}_n(\xi) = \sqrt{\frac{2n-3}{2}} \int_{-1}^{\xi} L_{n-2}(t)\, dt$$

$$= \sqrt{\frac{2n-3}{2}} \, \frac{1}{2n-3} \int_{-1}^{\xi} L'_{n-1} - L'_{n-3} \, dt$$

$$= \sqrt{\frac{2n-3}{2}} \, \frac{1}{2n-3} \left(L_{n-1}(\xi) - L_{n+1}(\xi) - L_{n-1}(-1) + L_{n+1}(-1) \right)$$

By induction, one can prove that

$$L_n(-1) = (-1)^n \quad \left(\text{with } L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x) \right)$$

$$\hat{b}_n(\xi) = \sqrt{\frac{2n-3}{2}} \, \frac{1}{2n-3} \left(L_{n-1}(\xi) - L_{n-3}(\xi) \right)$$

$$= \frac{1}{\sqrt{2(2n-3)}} \left(L_{n-1}(\xi) - L_{n-3}(\xi) \right)$$

First it is clear that $\hat{M}$ is symmetric:
the we compute the upper part:

$$\int_{-1}^{1}(\hat{b}_1)^2(\xi)\,d\xi = \int_{-1}^{1}\left(\frac{1-\xi}{2}\right)^2 d\xi = \int_{-1}^{1}\frac{\xi^2-2\xi+1}{4}\,d\xi$$

$$= \frac{2/3+2}{4} = \frac{2}{3}$$

$$\int_{-1}^{1}\hat{b}_1\,\hat{b}_2 = \int_{-1}^{1}\left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right)d\xi = \int_{-1}^{1}\frac{1-\xi^2}{\cancel{2}\,4} = \frac{2-2/3}{4} = \frac{1}{3}$$

$$\int_{-1}^{1}\hat{b}_1\,\hat{b}_3 = \int_{-1}^{1}\left(\frac{L_0-L_1}{2}\right)\frac{1}{\sqrt{6}}\left(L_2-L_0\right)$$

Due to the orthogonality relation, one has:

$$\int\hat{b}_1\hat{b}_3(\xi) = -\int_{-1}^{1}\frac{1}{2\sqrt{6}}\,L_0^2(\xi)\,d\xi = -\frac{1}{\sqrt{6}}$$

$$\int_{-1}^{1}\hat{b}_1(\xi)\,\hat{b}_4(\xi)\,d\xi = \int_{-1}^{1}\left(\frac{L_0(\xi)-L_1(\xi)}{2}\right)\frac{1}{\sqrt{10}}\left(L_3(\xi)-L_1(\xi)\right)d\xi$$

$$= \frac{1}{2\sqrt{10}}\int_{-1}^{1}\left(L_1(\xi)\right)^2 d\xi = \frac{1}{2\sqrt{10}}\cdot\frac{2}{3} = \frac{1}{3\sqrt{10}}.$$

For $n \geqslant 5$, one has:

$$\int_{-1}^{1}\hat{b}_1(x)\,\hat{b}_n(x)\,dx = \int_{-1}^{1}\left(\frac{L_0(\xi)-L_1(\xi)}{2}\right)\frac{1}{\sqrt{2(n-3)+2}}\left(L_{n-1}(\xi)-L_{n-3}(\xi)\right)$$

As $n-1>1$ and $n-3>1$, we have:

$$\int_{-1}^{1}\hat{b}_1(\xi)\,\hat{b}_n(\xi)\,d\xi = 0.$$

$$\int_{-1}^{1}\left(\hat{b}_2(\xi)\right)^2 = \int_{-1}^{1}\left(\frac{1+\xi}{2}\right)^2 d\xi = \int_{-1}^{1}\frac{\xi^2+2\xi+1}{4}\,d\xi$$

$$= \frac{2/3+0+2}{4} = \frac{2}{3}.$$

$$\int_{-1}^{1}\hat{b}_2(\xi)\,\hat{b}_3(\xi)\,d\xi = \int_{-1}^{1}\left(\frac{L_0(\xi)+L_1(\xi)}{2}\right)\left(\frac{L_2(\xi)-L_0(\xi)}{\cancel{\ }}\right)\frac{1}{\sqrt{6}}\,d\xi$$

$$= \int_{-1}^{1} -\frac{1}{\sqrt{6}}\,\frac{L_0^2(\xi)}{\cancel{4}2}\,d\xi = \frac{-1}{\sqrt{6}}.$$

$$\int_{-1}^{1}\hat{b}_2(\xi)\,\hat{b}_4(\xi) = \int_{-1}^{1}\left(\frac{L_0(\xi)+L_1(\xi)}{2}\right)\frac{1}{\sqrt{10}}\left(L_3(\xi)-L_1(\xi)\right)d\xi$$

$$= \int_{-1}^{1} -\frac{1}{2\sqrt{10}}\,L_1^2(\xi)\,d\xi = -\frac{1}{3\sqrt{10}}$$

For $n \geqslant 5$,

$$\int_{-1}^{1}\hat{b}_2(\xi)\,\hat{b}_n(\xi)\,d\xi = \int_{-1}^{1}\left(\frac{L_0(\xi)+L_1(\xi)}{2}\right)\frac{1}{\sqrt{2(2n-3)}}\left(L_{n-1}(\xi)-L_{n-3}(\xi)\right)d\xi$$

As $n-1>1$ and $n-3>1$, we have:

$$\int_{-1}^{1}\hat{b}_2(\xi)\,\hat{b}_n(\xi)\,d\xi = 0.$$

For $n \geq 3$, we have:

$$\int_{-1}^{1} \left[\hat{b}_n(\xi)\right]^2 d\xi = \int_{-1}^{1} \frac{1}{2(2n-3)} \left(L_{n-1}(\xi) - L_{n-3}(\xi)\right)^2 d\xi$$

$$= \frac{1}{2(2n-3)} \int_{-1}^{1} \left[L_{n-1}(\xi)\right]^2 - 2 L_{n-1}(\xi) L_{n-3}(\xi) + \left[L_{n-3}(\xi)\right]^2 d\xi.$$

$$= \frac{1}{2(2n-3)} \int_{-1}^{1} \left[L_{n-1}(\xi)\right]^2 + \left[L_{n-3}(\xi)\right]^2 d\xi$$

$$= \frac{1}{2(2n-3)} \left[\frac{1}{2n-1} + \frac{1}{2n-5}\right] 2$$

$$= \frac{2(2n-3)}{(2n-3)(2n-1)(2n-5)} = \frac{2}{(2n-1)(2n-5)}.$$

$$\int_{-1}^{1} \hat{b}_n(\xi)\, \hat{b}_{n+1}(\xi)\, d\xi = \frac{1}{\sqrt{2(2n-3)}\sqrt{2(2n-1)}} \int_{-1}^{1} \left(L_{n-1}(\xi) - L_{n-3}(\xi)\right)\left(L_n(\xi) - L_{n-2}(\xi)\right)$$

$$= 0.$$

$$\int \hat{b}_n(\xi)\, \hat{b}_{n+2}(\xi)\, d\xi = \frac{1}{\sqrt{2(2n-3)}} \frac{1}{\sqrt{2(2n+1)}} \int_{-1}^{1} \left(L_{n-1}(\xi) - L_{n-3}(\xi)\right)\left(L_{n+1}(\xi) - L_{n-1}(\xi\right.$$

$$= \frac{-1}{2\sqrt{(2n+1)(2n-3)}} \int \left[L_{n-1}(\xi)\right]^2 d\xi$$

$$= \frac{-1}{2\sqrt{(2n+1)(2n-3)}} \frac{2}{2n-1} = \frac{-1}{(2n-1)\sqrt{(2n+1)(2n-3)}}$$

For $m > n+2$

$$\int \hat{b}_n \hat{b}_m(\xi) = \frac{1}{2\sqrt{2n-3}\sqrt{2m-3}} \int_{-1}^{1} \left(L_{n-1}(\xi) - L_{n-3}(\xi)\right)\left(L_{m-1}(\xi) - L_{m-3}(\xi)\right) d\xi$$

as $n-1 \neq m-3$, $n-1 \neq m-1$, $n-3 \neq m-3$, $n-3 \neq m$

$$\int \hat{b}_n(\xi)\, \hat{b}_m(\xi)\, d\xi = 0.$$

b) $\hat{b}_1(\xi) = \frac{1-\xi}{2}$, $(\hat{b}_1)'(\xi) = -\frac{1}{2} = -\frac{L_0(\xi)}{2}$

$\hat{b}_2(\xi) = \frac{1+\xi}{2}$, $(\hat{b}_2)'(\xi) = \frac{1}{2} = \frac{L_0(\xi)}{2}$

$(n \geq 3)$ $\hat{b}_n(\xi) = \sqrt{\frac{2n-3}{2}} \int_{-1}^{\xi} L_{n-2}(\xi)\, d\xi$

$(\hat{b}_n)'(\xi) = \sqrt{\frac{2n-3}{2}}\, L_{n-2}(\xi)$

$$\int_{-1}^{1} (\hat{b}_1')^2 = \int_{-1}^{1} \left(-\frac{1}{2}\right)^2 d\xi = \frac{1}{2}$$

$$\int_{-1}^{1} \hat{b}_1'(\xi)\, \hat{b}_2'(\xi)\, d\xi = \int_{-1}^{1} -\frac{1}{2}\,\frac{1}{2}\, d\xi = -\frac{1}{2}.$$

For $n \geq 3$

$$\int_{-1}^{1} \hat{b}_1'(\xi)\hat{b}_n'(\xi) = \sqrt{\frac{2n-3}{2}} \int_{-1}^{1} -\frac{L_0(\xi)}{2}\, L_{n-2}(\xi)\, d\xi$$

As $n-2>0$, one has:

$$\int \hat{b}_1'(\xi)\, \hat{b}_n'(\xi)\, d\xi = 0 \qquad \text{for } n \geq 3$$

for $n \geq 3$, one has:

$$\int_{-1}^{1} [\hat{b}_n'(\xi)]^2\, d\xi = \frac{2n-3}{2} \int_{-1}^{1} [L_{n-2}(\xi)]^2\, d\xi$$

$$= \frac{2n-3}{2}\, \frac{2}{2n-3} = 1$$

For $m > n \geq 3$ we have:

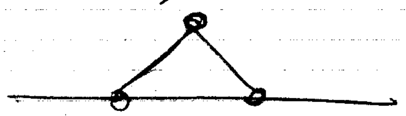
$$\int \hat{b}_n'(\xi)\,\hat{b}_m'(\xi)\,d\xi = \sqrt{\frac{(2n-3)(2m-3)}{2}} \int_{-1}^{1} L_{n-2}(\xi)\,L_{m-2}(\xi)\,d\xi$$

As $n-2 \neq m-2$, we have:

$$\int \hat{b}_n'(\xi)\,\hat{b}_m'(\xi)\,d\xi = 0.$$

d) First, if we restrict the global shape function to K_i, we remark that we have P polynomials of degree smaller or equal to $P-1$.

Moreover, as $\hat{b}_1, \dots, \hat{b}_p$ spans $\mathbb{P}_{p-1}$. We have only to check the continuity for $b_N^{i(p-1)+j}$, we have the standard



shape function.

For the other the continuity comes from

for $p \geq 3$,

$$b_p(-1) = \sqrt{\frac{2n-3}{2}} \int_{-1}^{-1} L_{n-2}(\xi)\,d\xi = 0$$

$$b_p(1) = \sqrt{\frac{2n-3}{2}} \int_{-1}^{1} L_{n-2}(\xi)\,L_0(\xi)\,d\xi = 0.$$