Serie 6:

Exercise 1:

(i) $D\phi(\xi) = \left[\, \xi_2(a_1 - a_4) + (1-\xi_2)(a_2 - a_3) \,\middle|\, \xi_1(a_1 - a_2) + (1-\xi_1)(a_4 - a_3) \,\right]$

$$\begin{aligned} \operatorname{Det} D\phi(\xi) = {} & \xi_1 \xi_2 \det(a_1 - a_4 | a_1 - a_2) \\ & + \xi_1(1-\xi_2) \det(a_2 - a_1 | a_2 - a_3) \\ & + (1-\xi_1)(1-\xi_2) \det(a_3 - a_2 | a_3 - a_4) \\ & + (1-\xi_1)\, \xi_2 \det(a_4 - a_3 | a_4 - a_1). \end{aligned}$$

(ii) $Au = \left(\sum_{j=1}^{N} A_{ij} u_j\right)_{j=1}^{N} \Longrightarrow \|Au\|^2 = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} A_{ij} u_j\right)^2$

By Cauchy Schwartz, one has:

$$\|Au\|^2 \leqslant \sum_{i=1}^{N} \left(\left(\sum_{j=1}^{N} A_{ij}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N} u_j^2\right)^{\frac{1}{2}} \right)^2 = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} A_{ij}^2 \cdot \|u\|^2 \right)$$
$$\leqslant \left(\sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij}^2 \right) \|u\|^2$$

This implies: $\|Au\| \leqslant \|A\|_F \, \|u\|$.

(iii) $\hat{u}(\hat{\xi}) = u(\phi(\hat{\xi})) \Longrightarrow u(\xi) = \hat{u}(\psi(\xi))$

This implies: $\nabla u(\xi) = D\psi(\xi) . \hat{\nabla} \hat{u}(\psi(\xi)) \Rightarrow \|\nabla u\|^2 \leq \|D\psi(\xi)\|^2 \times \|\hat{\nabla}\hat{u}(\psi(\xi))\|$

Hence, one has: $\|\nabla u(\xi)\|_2^2 \leqslant \|D\psi(\xi)\|_F^2 \, \|\hat{\nabla}\hat{u}(\psi(\xi))\|_2^2$.

$$\int_K \|\nabla u(\xi)\|^2 d\xi \leqslant \int_K \|D\psi(\xi)\|_F^2 \cdot \|\hat{\nabla}\hat{u}(\psi(\xi))\|_2^2 \, d\xi$$
$$\leqslant \int_{\hat{K}} \|D\psi(\phi(\hat{\xi}))\|_F^2 \cdot |\operatorname{Det} D\phi(\hat{\xi})| \, \|\nabla\hat{u}(\hat{\xi})\|_2^2 \, d\hat{\xi}$$
$$\leqslant \max_{\hat{\xi} \in \hat{K}} \left[\|D\psi(\phi(\hat{\xi}))\|_F^2 \, |\operatorname{Det} D\phi(\hat{\xi})| \right] \cdot \int_{\hat{K}} \|\hat{\nabla}\hat{u}(\hat{\xi})\|^2 d\hat{\xi}.$$

Finally, we have:

$$|u|_{H^1(K)} \leq \max_{\hat{\xi} \in \hat{K}} \left[\frac{\| D\Psi(\phi(\hat{\xi}))\|_F}{|\mathrm{Det}\, D\Psi(\phi(\hat{\xi}))|^{1/2}} \right] \cdot |\hat{u}|_{H^1(\hat{K})}.$$

$$\leq \max_{\hat{\xi} \in \hat{K}} \left[\frac{\| D\Psi(\xi)\|_F}{|\mathrm{Det}\, D\Psi(\xi)|} \right] \cdot |\hat{u}|_{H^1(\hat{K})}.$$

(iv) We denote by $D^2 u$ the matrix

$$\begin{bmatrix} \frac{\partial^2 u}{\partial \xi_1^2} & \frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 u}{\partial \xi_2^2} \end{bmatrix} =: D^2 u$$

Then, one has:

$$\hat{D}^2 \hat{u}(\hat{\xi}) = \left(D\phi(\hat{\xi}) \right) . \left(D\phi(\hat{\xi}) \right) \; D^2 u \left(\phi(\hat{\xi}) \right)$$

$$\| \hat{D}^2 \hat{u} \|_F (\hat{\xi}) \leq \| D\phi \|_F^2 (\hat{\xi}) \, \| D^2 u \|_F (\phi(\hat{\xi}))$$

This implies:

$$\int_{\hat{K}} \| \hat{D}^2 \hat{u} \|_F^2 (\hat{\xi}) \leq \int_K \| D\phi \|_F^4 (\Psi(\xi)) \cdot |\det D\Psi(\xi)| \cdot \| D^2 u \|_F^2 (\xi)$$

We remark that:

$$|\hat{u}|^2_{H^2(\hat{K})} = \int_{\hat{K}} \| \hat{D}^2 \hat{u} \|_F^2 (\hat{\xi})$$

$$|u|^2_{H^2(K)} = \int_K \| D^2 u \|_F^2 (\xi)$$

$$|u|_{H^2(\hat{K})} \leq \max_{\hat{\xi} \in \hat{K}} \left[\frac{\| D\phi(\hat{\xi}) \|_F^2}{|\det D\phi(\hat{\xi})|^{1/2}} \right] \; |\hat{u}|_{H^1(K)}$$

(v)

$$|u - Qu|_{H^1(K)} \leq |u - p - Q(u-p)|_{H^1(K)} \quad \text{mit } p/\ \hat{p} \in Q_1$$

$$\leq \max_{\xi \in K} \left[\frac{\| D\Psi(\xi) \|_F}{|\mathrm{Det}\, D\Psi(\xi)|^{1/2}} \right] \left| \hat{u} - \hat{p} - \hat{Q}(\hat{u} - \hat{p}) \right|_{H^1(K)}$$

$$\leq C_1 \max_{\xi \in K} \left[\frac{\| D\Psi(\xi) \|_F}{|\mathrm{Det}\, D\Psi(\xi)|^{1/2}} \right] \| \hat{u} - \hat{p} \|_{H^1(K)}, \quad C_1 = \| Id - \hat{Q} \|$$

$$\leq C \gamma \max_{\xi \in K} [-] \left(\sum_{k=1}^{2} \left\| \frac{\partial^2 \hat{u}}{\partial \hat{\xi}_k} \right\|_{L^2}^2 \right)^{1/2}$$

$$\leq C \gamma \max_{\xi \in K} \left[\frac{\| D\Psi(\xi) \|_F}{|\det D\Psi(\xi)|^{1/2}} \right] \times \max_{\hat{\xi} \in \hat{K}} \left[\frac{\| D^2 \phi(\hat{\xi}) \|_F^2}{|\det D\phi(\hat{\xi})|^{1/2}} \right] \| u \|_{H^2(K)}$$

Exercise 2:

For $\Omega = \mathbb{R}^2$, one has for all $u \in H^2(\mathbb{R}^2)$

$$\|u\|^2_{H^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \nabla u^2 + u^2 = \int_{\mathbb{R}^2} -\Delta u \, u + u^2$$
$$\leq \left(\|\Delta u\|_{L^2(\mathbb{R}^2)} \|u\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)} \right)$$
$$\leq C_1 \|u\|_{H^2(\mathbb{R}^2)} \|u\|_{L^2(\mathbb{R}^2)} .$$

For Ω a Lipschitz domain, forall $u \in H^2(\Omega)$
$\exists\, \tilde{u} \in H^2(\mathbb{R}^2) \;/\; \tilde{u} = u$ in Ω and

$$\|\tilde{u}\|_{L^2(\mathbb{R}^2)} \leq C \|u\|_{L^2(\Omega)},$$
$$\|\tilde{u}\|_{H^1(\mathbb{R}^2)} \leq C \|u\|_{H^1(\Omega)},$$
$$\|\tilde{u}\|_{H^2(\mathbb{R}^2)} \leq C \|u\|_{H^2(\Omega)} .$$

This implies:

$$\|u\|^2_{H^1(\Omega)} \leq \|\tilde{u}\|^2_{H^1(\mathbb{R}^2)} \leq C \|\tilde{u}\|_{L^2(\mathbb{R}^2)} \|\hat{u}\|_{H^2(\mathbb{R}^2)}$$
$$\leq C \|u\|_{L^2(\Omega)} \|u\|_{H^2(\Omega)} .$$

Exercise 3:

(i) $$\int_{\tilde K}\left[\tilde u(\tilde x,\tilde y)\right]^2 d\tilde x\, d\tilde y = \int_{\tilde K}\left[u(\tilde x,\delta\tilde y)\right]^2 d\tilde x\, d\tilde y$$

$$= \frac{1}{\delta}\int_{\tilde K}\left[u(\tilde x,\delta\tilde y)\right]^2 d\tilde x\,(k\, d\tilde y)$$

$$= \frac{1}{\delta}\int_K\left[u(x,y)\right]^2 dx\, dy$$

$$\|\tilde u\|_{L^2(\tilde K)} = \frac{1}{\sqrt{\delta}}\|u\|_{L^2(K)}.$$

(ii) $$\int_{\tilde K}\left[\frac{\partial^2\tilde u}{\partial\tilde x^2}(\tilde x,\tilde y)\right]^2 d\tilde x\, d\tilde y = \int_{\tilde K}\left[\frac{\partial^2 u}{\partial x^2}(\tilde x,\delta\tilde y)\right]^2 \frac{1}{\delta}\, d\tilde x\,(\delta d\tilde y)$$

$$= \frac{1}{\delta}\int_K \frac{\partial^2 u}{\partial x^2}(x,y)\, dx\, dy$$

$$\left\|\frac{\partial^2\tilde u}{\partial\tilde x^2}\right\|_{L^2(\tilde K)} = \frac{1}{\sqrt{\delta}}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^2(K)}.$$

$$\int_{\tilde K}\left[\frac{\partial^2\tilde u}{\partial\tilde x\partial\tilde y}(\tilde x,\tilde y)\right]^2 d\tilde x\, d\tilde y = \int_{\tilde K}\left[\frac{\partial^2 u}{\partial x\partial y}(\tilde x,\delta\tilde y)\cdot\delta\right]^2\frac{1}{\delta}\, d\tilde x\,(\delta d\tilde y)$$

$$= \delta\int_K\left[\frac{\partial^2 u}{\partial x\partial y}(x,y)\right]^2 dx\, dy$$

$$\left\|\frac{\partial^2 u}{\partial\tilde x\partial\tilde y}\right\|_{L^2(\tilde K)} = \sqrt{\delta}\left\|\frac{\partial^2 u}{\partial x\partial y}\right\|_{L^2(K)}.$$

$$\int_{\tilde K}\left[\frac{\partial^2\tilde u}{\partial\tilde y^2}\right]^2 d\tilde x\, d\tilde y = \int_{\tilde K}\left[\frac{\partial^2 u}{\partial y^2}(\tilde x,\delta\tilde y)\,\delta^2\right]^2\frac{1}{\delta}\, d\tilde x\,(\delta d\tilde y)$$

$$= \delta^3\int_{\tilde K}\left[\frac{\partial^2 u}{\partial y^2}(x,y)\right]^2 dx\, dy$$

$$\left\|\frac{\partial^2 u}{\partial\tilde y^2}\right\|_{L^2(\tilde K)} = \delta^{3/2}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2(K)}.$$

(ii) $|\tilde{K}| = \frac{1}{2}$, $|\hat{K}| = \frac{1}{2}$,

Radius of inscribed circle:

$$\tilde{r}\,(a+b+c) = \frac{1}{2}, \quad \hat{r}\,(a+b+c) = \frac{1}{2}$$

$$\frac{1}{8} \leq \tilde{r} \leq \frac{1}{4} \quad \text{and} \quad \frac{1}{8} \leq \hat{r} \leq \frac{1}{4}$$

diameter of $\tilde{K}$ and $\hat{K}$:

$$1 \leq \tilde{e} \leq \sqrt{2}, \quad \hat{e} = \sqrt{2}.$$

Thus implies:

$$\begin{cases} \underline{c}\ \|\tilde{u}\|_{L^2(\tilde{K})} \leq \|\hat{u}\|_{L^2(\hat{K})} \leq \bar{c}\ \|\tilde{u}\|_{L^2(\tilde{K})}, \\ \underline{c}\ |\tilde{u}|_{H^2(\tilde{K})} \leq |\hat{u}|_{H^2(\hat{K})} \leq \bar{c}\ \|\tilde{u}\|_{L^2(\hat{K})}. \end{cases}$$

(iii) This comes from the last inequality and the Bramble-Hilbert lemma

$$\|u - I_{1,K}u\|_{L^2(K)} \leqslant C\sqrt{\delta}\,\|\hat{u} - I_{1,\hat{K}}\hat{u}\|_{L^2(\hat{K})}$$

As for all $\hat{p} \in \mathcal{P}_1$ $\hat{p} = I_{1,\hat{K}}\hat{p}$:

$$\|u - I_{1,\hat{K}}u\|_{L^2(K)} \leqslant C\sqrt{\delta}\,\|\hat{u} - \hat{p} - I_{1,\hat{K}}(\hat{u} - \hat{p})\|_{L^2(\hat{K})}$$
$$\leqslant C\sqrt{\delta}\,\|Id - I_{1,\hat{K}}\|\,\|\hat{u} - \hat{p}\|_{L^2(\hat{K})}$$
$$\leqslant C\sqrt{\delta}\,|\hat{u}|_{H^2(\hat{K})}$$
$$\leqslant C\sqrt{\delta}\,|\tilde{u}|_{H^2(\tilde{K})}.$$

The

$$|\tilde{u}|^2_{H^2(\tilde{K})} = \left\|\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}\right\|^2_{L^2(\tilde{K})} + \left\|\frac{\partial^2\tilde{u}}{\partial x\partial y}\right\|^2_{L^2(\tilde{K})} + \left\|\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right\|^2_{L^2(\tilde{K})}$$
$$= \frac{1}{\delta}\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L^2(K)} + \delta^2\left\|\frac{\partial^2 u}{\partial x\partial y}\right\|^2_{L^2(K)} + \delta^4\left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{L^2(K)}\right)$$
$$\leqslant \frac{C}{\delta}\,|u|^2_{H^2(K)}.$$

$$\Rightarrow \|u - I_{1,K}u\|_{L^2(K)} \leqslant C\,|u|_{H^2(K)}.$$

(iv) Let h be the larger side of a triangle. By scaling the coordinates

$$x = h\hat{x}, \quad y = h\hat{y}$$

we have for any triangle

$$\|u - I_{1,K} u\|_{L^2(K)} \leq C_1 h^2 |u|_{H^2(K)}$$

Let h_M be the maximum of all the side of the triangle of the mesh.

$$\forall K \in \mathcal{M} \quad \|u - I_{1,K} u\|^2_{L^2(K)} \leq C h^4 |u|^2_{H^2(K)}$$

$$\Rightarrow \|u - I_{1,M} u\|^2_{L^2(\Omega)} \leq \sum_{K \in \mathcal{M}} \|u - I_{1,K} u\|^2_{L^2(K)}$$

$$\leq C h^4 \sum_{K \in \mathcal{M}} |u|_{H^2(K)}$$

$$\leq C h^4 |u|^2_{H^2(\Omega)}$$

$$\|u - I_{1,M} u\|_{L^2(\Omega)} \leq C h^2 |u|_{H^2(\Omega)}.$$