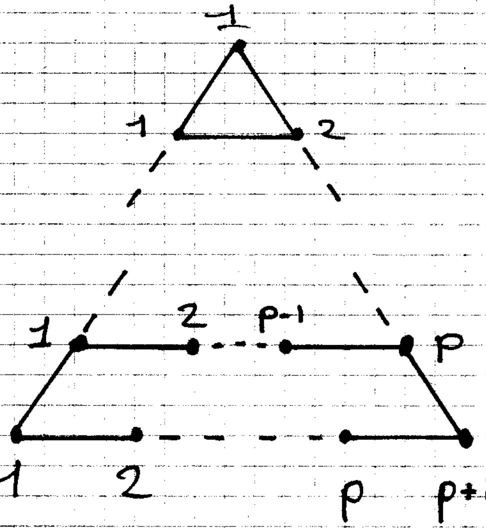
Serie 5

Exercise 1:

to compute the dimension of $\mathcal{P}_p(K) \cap H_0^1(K)$, it is usefull to introduce the following degrees of freedom:

$$\beta_K^i(v) = v(a^i)$$

with a^i one of these points:



the cardinal of $\{\beta_K^i\}$ is given by

$$(p+1) + p + \dots + 2 + 1 = \frac{(p+2)(p+1)}{2}.$$

$$\Rightarrow \dim \mathcal{P}_p(K) = \frac{(p+2)(p+1)}{2} = \frac{p^2+3p+2}{2}$$

To ensure $v|_{\partial K} = 0$, it is enough to set

$\beta_K^i(v) = 0$ for all i such that a^i is on the boundary ∂K.

The number of these (a^i)'s is $3p$

This implies that $\dim\left[\mathcal{P}_p(K) \cap H_0^1(K)\right] = \frac{p^2+3p+2}{2} - 3p = \frac{(p-2)(p-}{2}$

Exercise 2:

$u \in [P^1(K)]^2$ is equivalent to:

$$u = \left| \begin{array}{l} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \end{array} \right.$$

Then, one has:

$$\mathrm{div}(u) = a_1 + b_2 \quad \in \mathcal{P}_0(K)$$

$\Rightarrow \mathcal{P}_1(K) = \{ v \in P^1(K) \mid \mathrm{div}(v) \in \mathcal{P}_0(K) \}$.
($\mathrm{div}\, v \in \mathcal{P}_0(K)$ is not an additional constraint).

A basis for this space is just:

$$\{(\lambda_i ; 0)\}_{i=1}^{3} \cup \{(0 ; \lambda_i)\}_{i=1}^{3}$$

where λ_i are the barycentric coordinates

Exercise 3)

a)

$$-\Delta u = f \text{, in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

$$\forall v \in H_0^1(\Omega) \quad -\int_\Omega \Delta u \, v = \int_\Omega f v$$

By integration by part, one has:

$$\int_\Omega \nabla u \, \nabla v - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v}_{=0} = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega).$$

This implies that we have to solve the problem:

$$(*) \quad \begin{cases} \text{Find } u \in V \text{ such that for all } v \in V: \\ a(u,v) = f(v) \end{cases}$$

with

$$\begin{cases} a(u,v) = \int_\Omega \nabla u \, \nabla v & V \times V \longrightarrow \mathbb{R} \\ f(v) = \int_\Omega f v & V \longrightarrow \mathbb{R} \\ V = H_0^1(\Omega) \end{cases}$$

<u>a is continuous:</u>

$$a(u,v) \leqslant \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leqslant \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

<u>a is V-elliptic:</u>

$$a(u,u) = \int |\nabla u|^2$$

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$$

Due to Poincaré inequality, one has:

$\exists C>0, \quad \|u\|^2_{L^2(\Omega)} \leqslant C \; \|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in H^1_0(\Omega)$

This leads to:

$$\|u\|^2_{H^1(\Omega)} \leqslant (1+C) \; \|\nabla u\|^2_{L^2(\Omega)} \leqslant (1+C) \; a(u,u)$$

~~Due~~ f is continuous:

$$f(v) = \int_\Omega f \, v \;\leqslant \|f\|_{L^2(\Omega)} \; \|v\|_{L^2(\Omega)} \leqslant C \; \|v\|_{H^1(\Omega)}$$

Hence, the problem $(*)$ is well-posed.

b) As $V_N \subset V$ and due to the V-ellipticity, ~~one~~ the discreet problem is well-posed.

c) Let E be a linear space such that $\dim E = N < +\infty$

if $\{l_i\}_{i=1}^{N}$ is a system of linear forms on E satisfying:

$$\{ l_i(v) = 0, \quad \forall i = 1, \dots, N \implies v = 0 \}$$

then $\{l_i\}_{i=1}^{N}$ is a basis of E'.

<u>Proof</u>: Let $\{v_n\}_{n=1}^{N}$ be a basis of E. All $v \in E$ can be written as: $\quad v = \sum_{n=1}^{N} d_n v_n$.

On this basis, the condition $l_i(v) = 0, \quad \forall i = 1, \dots, N$ becomes $\sum_{n=1}^{N} d_n l_i(v_n) = 0$.

If we define the matrix $A = [l_i(v_n)]_{i,n=1}^{N}$ and $b = [d_n]_{n=1}^{N}$.

$$\{ l_i(v) = 0, \quad \forall i = 1, \dots, N \iff Ab = 0 \} \Rightarrow \{ v = 0 \iff b = 0 \}$$

Then $Ab = 0 \Rightarrow b = 0$ means that A is injective and therefore bijective

$$A \text{ bijective} \implies A^H \text{ bijective}$$

which implies that:

$$\{ A^T A = 0 \implies A = 0 \} \text{ i.e. } \{ \sum_{i=1}^{N} \lambda_i l_i(v_n) = 0 \; \forall n \implies \lambda_i = 0 \; \forall i \}$$

which implies: $\sum \lambda_i l_i = 0 \implies \lambda_i = 0$

Hence, the family $\{l_i\}_{i=1}^{N}$ is free and is therefore a basis.

c) The space $\mathcal{P}_3(K)$ is spanned by:

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$$

This implies that:

$$\dim \mathcal{P}_3(K) = 10.$$

We have to prove that (as card $\{\beta_K^j\}_{j=1}^{10} = 10$)

$$\beta_K^j(v) = 0, \qquad \forall j = 1, \dots, 10 \quad \Rightarrow \quad v = 0$$

Using the barycentric coordinates $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x) = 1 - \lambda_1(x) - \lambda_2(x)$

$$v \in \mathcal{P}_3(K) \iff v = \hat{p}_1(\lambda_2) + \lambda_1 \hat{q}_1(\lambda_1, \lambda_2)$$

with $p_1 \in \mathcal{P}_3([0,1])$ and $q_1 \in \mathcal{P}_2(\hat{K})$.

$$\begin{cases} \beta_K^2(x) = 0 \iff \hat{p}_1(1) = 0 \\ \beta_K^3(x) = 0 \iff \hat{p}_1(0) = 0 \\ \beta_K^5(x) = 0 \iff -\hat{p}_1'(1) = 0 \\ \beta_K^9(x) = 0 \iff \hat{p}_1'(0) = 0 \end{cases} \xRightarrow{\hat{p}_1 \in \mathcal{P}_3([0,1])} \hat{p}_1 \equiv 0$$

And therefore, one has: $v = \lambda_1 \hat{q}_1$ with $q_1 \in \mathcal{P}_2(K)$

In the same way, we have:

$$v = \lambda_1 q_1 = \lambda_2 q_2 = \lambda_3 q_3 \quad \text{with } q_1, q_2, q_3 \in \mathcal{P}_2(K)$$

This implies that:

$$v = \lambda_1 \lambda_2 \lambda_3 c \quad \text{with } c \in \mathcal{P}_0 \equiv \mathbb{R}.$$

Finally as $\beta_K^{10}(v) = v\left(\frac{a_1 + a_2 + a_3}{3}\right) = 0 \iff \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} c = 0 \iff c = 0$

Hence $v = 0$

c) For $i = 1, \dots, 3$ β_K^i is associated to node

For $i = 4, \dots, 9$ β_K^i is associated with edge

For $i = 10$ β_K^i is associated with interior point.

Série 5:

d) As $b_K^1 \in P_3(K)$, b_K^1 can be written as:

$$b_K^1(x) = p_1(\lambda_2(x)) + \lambda_1(x) \cdot q_1(\lambda_1(x), \lambda_2(x))$$

with $p_1 \in P_3([0;1])$ and $q_1 \in P_2(\hat{K})$ with $\hat{K}$ the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$

$0 = \beta_2^K(b_K^1) = b_K^1(a_2) = p_1(1) + 0 \cdot q_1(0,1)$

$0 = \beta_K^3(b_K^1) = b_K^1(a_3) = p_1(0) + 0 \; q_1(0,0)$

$0 = \beta_K^5(b_K^1) = \nabla b_K^1(a_2) \cdot (a_3 - a_2) = -p_1'(1) + 0 \cdot \frac{\partial q}{\partial y}(0,1)$

$0 = \beta_K^9(b_K^1) = \nabla b_K^1(a_3)(a_3 - a_2) = p_1'(0) + 0 \cdot \frac{\partial q}{\partial y}(0,0)$

As $p_1 \in P_3([0,1])$ and $p_1(0) = p_1(1) = p_1'(0) = p_1'(1) = 0$, we have $p_1 \equiv 0$

(1) $b_K^1(x) = \lambda_1(x) \, q_1(\lambda_1(x), \lambda_2(x))$ with $q_1 \in P_2(\hat{K})$.

Then, as $b_K^1 \in P_3(K)$, we have:

$$b_K^1(x) = p_2(\lambda_1(x)) + \lambda_2(x) \; q_2(\lambda_1(x), \lambda_2(x))$$

with $p_2 \in P_3([0;1])$ and $q_2 \in P_2(\hat{K})$.

$0 = b_K^1(a_3) = p_2(0)$, $1 = b_K^1(a_1) = p_2(1)$, $0 = (\nabla b_K^1(a_3)) \cdot (a_1 - a_3) = p_2'(0)$, $0 = -p_2'(1)$

As $p_2 \in P_3([0;1])$, this implies: $p_2(t) = 3t^2 - 2t^3$.

(2) $b_K^1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) + \lambda_2(x) \; q_2(\lambda_1(x), \lambda_2(x))$.

$$b_K^1(x) = p_3(\lambda_1(x)) + \lambda_3(x) \; q_3(\lambda_1(x), \lambda_3(x))$$

with $p_3 \in P_3([0,1])$, $p_3(0) = 0$, $p_3'(0) = 0$, $p_3(1) = 1$, $p_3'(1) = 0 \Rightarrow p_3(t) = 3t^2 - 2t^3$.

$q_3 \in P_2(\hat{K})$.

(3) $b_K^1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) + \lambda_3(x) \; q_3(\lambda_1(x), \lambda_3(x))$

$(2)+(3) \Rightarrow \quad \lambda_2(x)\, q_2(\lambda_1(x), \lambda_2(x)) = \lambda_3(x)\, q_3(\lambda_1(x), \lambda_3(x))$

$\Rightarrow \lambda_2(x)\, q_2(\lambda_1(x), \lambda_2(x)) = \lambda_2(x)\lambda_3(x)\, q(x), \; q \in P_1(K)$

Hence, one has: $b_K^1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) + \lambda_2(x)\lambda_3(x)\, q_4(x)$

$+(1) \Rightarrow b_K^1(x) = \lambda_1(x)\, q_1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) + \lambda_2(x)\lambda_3(x)\, q_4(x)$

$\Rightarrow \lambda_1(x)\left[q_1(x) + 2\lambda_1^2(x) - 3\lambda_1(x)\right] = \lambda_2(x)\lambda_3(x)\, q_4(x)$

$\Rightarrow q_4(x) = \lambda_1(x)\, q_5(x)$ with $q_5 \in P_0(K)$

$= c\, \lambda_1(x),$ with $c \in \mathbb{R}$.

$b_K^1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) + c\,\lambda_1(x)\lambda_2(x)\lambda_3(x).$

To conclue, we ensure $b_K^1\left(\frac{a_1+a_2+a_3}{3}\right) = 0.$

$0 = \frac{3}{9} - \frac{2}{27} + \frac{c}{27} \Rightarrow c = -7.$

$$\boxed{b_K^1(x) = 3\lambda_1^2(x) - 2\lambda_1^3(x) - 7\lambda_1(x)\lambda_2(x)\lambda_3(x)}$$

By permutation, we have:

$$\boxed{b_K^2(x) = 3\lambda_2^2(x) - 2\lambda_2^3(x) - 7\lambda_1(x)\lambda_2(x)\lambda_3(x)}$$

$$\boxed{b_K^3(x) = 3\lambda_3^2(x) - 2\lambda_2^3(x) - 7\lambda_1(x)\lambda_2(x)\lambda_3(x)}$$

Same technique for $b_K^4(x)$

$$b_K^4(x) = p_1(\lambda_2(x)) + \lambda_1(x)\, q_1(x), \quad p_1 \in P^3([0,1]);\ q_1 \in P^2(K)$$
$$= p_2(\lambda_1(x)) + \lambda_2(x)\, q_2(x), \quad p_2 \in P^3([0,1]),\ q_2 \in P^2(K)$$
$$= p_3(\lambda_1(x)) + \lambda_3(x)\, q_3(x), \quad p_3 \in P^3([0,1]),\ q_3 \in P^2(K)$$

with $p_1(0)=0,\ p_1'(0)=0,\ p_1(1)=0,\ p_1'(1)=0, \Rightarrow p_1(t)=0$

$p_2(0)=0,\ p_2'(0)=0,\ p_2(1)=0,\ p_2'(1)=-1, \Rightarrow p_2(t)=t^2-t^3$

$p_3(0)=0,\ p_3'(0)=0,\ p_3(1)=0,\ p_3'(1)=0, \Rightarrow p_3(t)=0$

Hence, one has:

$$b_K^4(x) = \underset{(1)}{\lambda_1(x)\, q_1(x)} = \underset{(2)}{[\lambda_1(x)]^2 - [\lambda_1(x)]^3 + \lambda_2(x) q_2(x)} = \underset{(3)}{\lambda_3(x) q_3(x)}$$

This implies that ((1)+(3))

$$b_K^4(x) = \lambda_1(x)\ \lambda_3(x)\left[a\ \lambda_1(x) + b\, \lambda_2(x) + c\right]$$
$$= \lambda_1(x)\left[1-\lambda_1(x)-\lambda_2(x)\right]\left[a\lambda_1(x) + b\lambda_2(x) + c\right]$$
$$= \lambda_1(x)\left(1-\lambda_1(x)\right)\left(a\lambda_1(x)+c\right) + \lambda_2(x)\ q_4(x)$$

If we take into account (2), we obtain:

$$[\lambda_1(x)]^2 - [\lambda_1(x)]^3 = \lambda_1(x)\left(1-\lambda_1(x)\right)\left(a\lambda_1(x)+c\right)$$
$$\Rightarrow a = 1, \quad c = 0$$

$$b_K^4(x) = \lambda_1(x)\ \lambda_3(x)\ \left[\lambda_1(x) + b\,\lambda_2(x)\right]$$

$$b_K^4\left(\frac{a_1+a_2+a_3}{3}\right) = 0 = \frac{1}{3} \times \frac{1}{3} \times \left[\frac{1}{3} + \frac{b}{3}\right]$$

This implies $b=-1$.

$$b_K^4 = \lambda_1(x)\ \lambda_3(x)\left[\lambda_1(x) - \lambda_2(x)\right]$$

By permutation, one has:

$$
\begin{aligned}
b_K^4(x) &= \lambda_1(x)\,\lambda_3(x)\,[\lambda_1(x) - \lambda_2(x)], \\
b_K^5(x) &= \lambda_2(x)\,\lambda_1(x)\,[\lambda_2(x) - \lambda_3(x)], \\
b_K^6(x) &= \lambda_3(x)\,\lambda_2(x)\,[\lambda_3(x) - \lambda_1(x)], \\
b_K^7(x) &= \lambda_1(x)\,\lambda_2(x)\,[\lambda_1(x) - \lambda_3(x)], \\
b_K^8(x) &= \lambda_2(x)\,\lambda_3(x)\,[\lambda_2(x) - \lambda_1(x)], \\
b_K^9(x) &= \lambda_3(x)\,\lambda_1(x)\,[\lambda_3(x) - \lambda_2(x)].
\end{aligned}
$$

Same technique for

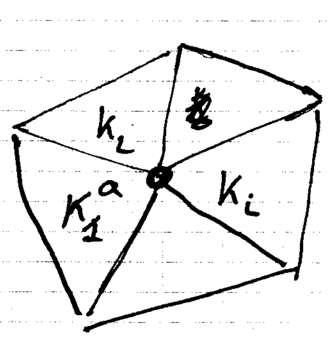
$$b_K^{10} = \lambda_1(x)\,p_1(x) = \lambda_2(x)\,p_2(x) = \lambda_3(x)\,p_3(x)$$

$$\Longrightarrow b_K^{10}(x) = c\,\lambda_1(x)\,\lambda_2(x)\,\lambda_3(x)$$

$$b_K^{10}\left(\frac{a_1+a_2+a_3}{3}\right) = c\,\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = 1 \qquad c = 27.$$

$$b_K^{10}(x) = 27\,\lambda_1(x)\,\lambda_2(x)\,\lambda_3(x).$$

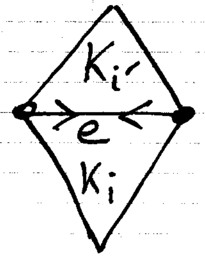
e) For β_K^j or b_K^j with $j = 1, \dots, 3$



Glueing the $b_{K_i}^{j_i}$ at the point a ($j_i \in \{1,2,3\}$) ensure for all $v \in V_H$

$\beta_{K_i}^{j_i}(v|_{K_i}) = \beta_{K_{i'}}^{j_{i'}}(v|_{K_{i'}})$.

$\Longrightarrow$ v is continuous at all the nodes of the mesh.



Glueing the 4 degrees of freedom relative to this edge ensure that $(v|_{K_i})|_e = (v|_{K_{i'}})|_{e'}$

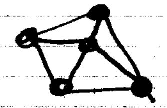
Indeed, in one D, this 4 degrees of freedom are enough to determine ~~the~~ a 1D polynom of P_3. (see 1D-hermitian finite element, serie 4).

$\Longrightarrow C^0$ on all Ω.

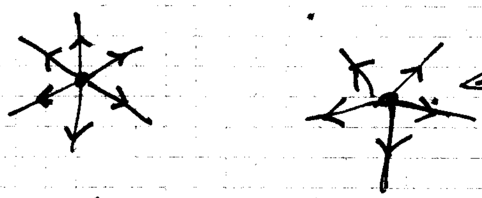
To ensure that $v_H|_{\partial\Omega} \equiv 0$, it is enough to ensure that all the degrees of freedom associated to edge or node of the boundary is 0!

f) For a Mesh $\mathcal{M}$ the global degrees of freedom are

① the value at each ~~points~~ node of the mesh

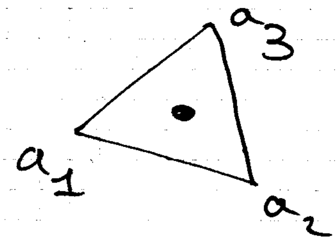


② the directional derivative at each node



5 degrees of freedom.

③ the value at the center of gravity of each triangle of the mesh.



Exercise 4:

Find $u \in V$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Sigma [v] \qquad \forall v \in V$$

For $v = 1$ one has

$$\int \nabla u \nabla 1 = 0 = \int_\Sigma [v]$$

Hence, one has for all $v \in H^1(\Omega \setminus \Sigma)$

$$\int_\Omega \nabla u \nabla v = \int_\Sigma v(0^+,.,.) - \int_\Sigma v(0^-,.,.)$$

For $v \in \mathcal{D}(\Omega \setminus \Sigma)$ we have

$$\int_\Omega \nabla u \nabla v = 0 \Rightarrow \int_\Omega -\Delta u\, v = 0$$

$$\Rightarrow \Delta u = 0, \quad in\ \Omega.$$

Then for $v \in H^1(\Omega \setminus \Sigma)$

$$\underbrace{-\int_\Omega \Delta u\, v}_{= 0} + \int_{\partial\Omega} \frac{\partial u}{\partial n} v + \int_\Sigma \frac{\partial v}{\partial x}(0^-,.) - \int_\Sigma \frac{\partial v}{\partial x}(0^+,.)$$

$$= \int_\Sigma v(0^+,.) - \int_\Sigma v(0^-,.)$$

$$\Rightarrow \frac{\partial u}{\partial n} = 0 \quad on \quad \partial\Omega$$

$$\frac{\partial u}{\partial x}(0,.) = 1$$

Finally, the boundary value problem is:

Find $u \in H^1(\Omega \setminus \Sigma)$ such that:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial x}(0,\cdot) = -1 & \text{on } \Sigma \\ \displaystyle\int_\Omega u = 0 & \end{cases}$$

Exercise 5:

(a) Remark:

if $p(\xi) \in \mathcal{P}_2([-1;1])$ with $p(-1) = a$, $p(1) = b$, $\int_{-1}^{1} \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) p(\xi)\, d\xi$

then $p(\xi) = a\,\frac{1-\xi}{2} + b\,\frac{1+\xi}{2} -$ [redacted] $\frac{15}{4}(1-\xi^2)$

b_K^1 : $\beta_K^1(b_K^1) = 1$ and $\beta_K^j(b_K^1) = 0$ for $j \neq 1$.

By definition, b_K^1 satisfies

$*$ $b_K^1(\varphi_K^1(-1)) = 0$; $b_K^1(\varphi_K^1(1)) = 0$

and $\int_{-1}^{1} b_K^1(\varphi_K^1(\xi)) \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) d\xi = 0$

This implies that

$$b_K^1(\varphi_K^1(\xi)) = 0 \qquad \forall \xi \in [-1;1]$$

$$\Updownarrow$$

$$b_K^1(x) = \lambda_1(x)\, q_1(x) \quad \text{with} \quad q_1(x) \in \mathcal{P}_1(K)$$

(1)

$$= (1 - \lambda_2(x) - \lambda_3(x))\, q_1(x)$$

$*$ $b_K^1(\varphi_K^2(-1)) = b_K^1(a_K^3) = 0,$

$b_K^1(\varphi_K^2(1)) = b_K^1(a_K^1) = 1$

$\int_{-1}^{1} b_K^1(\varphi_K^2(\xi)) \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) d\xi = 0$

This implies:

$$b^1_K(\gamma^2_K(\xi)) = \frac{1+\xi}{2}.$$

$$\Updownarrow$$

(2) $$b^1_K(x) = \lambda_1(x) + q_2(x) * \lambda_2(x) \quad \text{with } q_2 \in P_1(K)$$

* $b^1_K(\gamma^3_K(-1)) = b^1_K(a^1_K) = 1$

$b^1_K(\gamma^3_K(1)) = b^2_K(a^3_K) = 0$

$\int_{-1}^1 b^1_K(\gamma^3_K(\xi)) \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) d\xi$

$$\Rightarrow b^1_K(\gamma^3_K(\xi)) = \frac{1-\xi}{2}$$

$$\Updownarrow$$

(3) $$b^1_K(x) = \lambda_1(x) + q_3(x) \cdot \lambda_3(x) \quad q_3 \in P_1(K)$$

(2) + (3) $\overset{(q_3\lambda_3 = q_2\lambda_2)}{\Longrightarrow}$ $b^1_K(x) = \lambda_1(x) + \lambda_2\lambda_3 \cdot c$ (4)

with $c \in \mathbb{R}$

(4) + (1) $\Longrightarrow$ $\lambda_1(x)\, q_1(x) = \lambda_1(x) + \lambda_2(x)\lambda_3(x) \cdot c$

for $\lambda_1 = 0,\ \lambda_2 = \frac{1}{2},\ \lambda_3 = \frac{1}{2}$

$$\Rightarrow c = 0$$

$$b^1_K(x) = \lambda_1(x)$$

permutation, one has:

$$* \; b_K^2(x) = \lambda_2(x),$$

$$* \; b_K^3(x) = \lambda_3(x).$$

b_K^4: $\beta_K^4(b_K^4) = 1$, $\quad \beta_K^j(b_K^4) = 0$, $\quad \forall j \neq 4$.

By definition,

$$b_K^4(\varphi_K^2(-1)) = 0, \quad b_K^4(\varphi_K^2(1)) = 0$$

and $\int_{-1}^{1} b_K^4(\varphi_K^2(\xi)) \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) d\xi = 0$

$$\Longrightarrow b_K^4(\varphi_K^2(\xi)) = 0$$

We have:

(4) $\qquad b_K^4(x) = \lambda_2(x)\, q_2(x)$ with $q_2 \in P_1$

In the same way, we have:

(5) $\qquad b_K^4(x) = \lambda_3(x)\, q_3(x)$ with $q_3 \in P_3$

$$* \; b_K^4(\varphi_K^1(-1)) = b_K^1(a_K^2) = 0$$

$$b_K^4(\varphi_K^1(1)) = b_K^1(a_K^3) = 0$$

$$\int_{-1}^{1} b_K^4(\varphi_K^1(\xi)) \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) d\xi = 1$$

$$b_K^4(\varphi_K^1(\xi)) = -\frac{5}{4}(1-\xi^2)$$

e remark that

$$b_K^4(\mathcal{F}_K^1(\xi)) = -15 \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right)$$

$$= -15 \quad \lambda_2(\mathcal{F}_K^1(\xi)) \times \lambda_3(\mathcal{F}_K^1(\xi))$$

Hence, one has:

(6) $b_K^4(x) = -15 \ \lambda_2(x)\ \lambda_3(x) + \lambda_1(x)\ q_1(x)$

with

$q_1 \in P_1(K)$

Hence, (4) + (5) + (6) lead to:

$$b_K^4(x) = -15\ \lambda_2(x)\ \lambda_3(x)$$

$b_K^5(x) = -15\ \lambda_3(x)\ \lambda_1(x)$; $b_K^6(x) = -15\ \lambda_1(x)\ \lambda_2(x)$

b) It is enough to remark that

$\beta_K^1, \beta_K^2, \beta_K^6$ fix the value on the edge $[a_1, a_2]$

$\beta_K^2, \beta_K^3, \beta_K^4$ fix the value on the edge $[a_2, a_3]$

$\beta_K^3, \beta_K^1, \beta_K^5$ fix the value on the edge $[a_3, a_1]$

These value depends only of the value of v on these intervals.

$\Longrightarrow$ trace fixing property

c) comes from a) and b)