Serie 3.

Exercise 1:

* $\operatorname{div} C \operatorname{grad} u = f$
* $5\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = f$

$$2^2\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = f$$

We define X and Y such that:

$$\frac{\partial}{\partial X} = 2\frac{\partial}{\partial x} \; ; \; \frac{\partial}{\partial Y} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \qquad (*)$$

which implies:

$$\frac{\partial x}{\partial X} = 2 \; ; \; \frac{\partial y}{\partial X} = 0$$

$$\frac{\partial x}{\partial Y} = 1 \; ; \; \frac{\partial y}{\partial Y} = 1$$

One solution is the following:

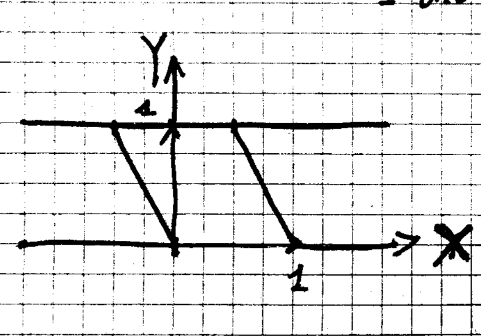
$$\begin{cases} x = 2X + Y \\ y = Y \end{cases} \iff \begin{cases} X = \frac{1}{2}[x - y] \\ Y = y \end{cases}$$

In $\hat{\Omega}_1 = \{(X,Y)\in \mathbb{R}^2 / Y \in]0;1[\text{ and } X \in]0-\frac{Y}{2};\frac{1}{2}-\frac{Y}{2}[$, $\phi^* u$ satisfies: (see *)

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\phi^* u(X,Y) = \left[\left(2\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2\right] u(\phi(X,Y))$$

$$= \operatorname{div} C \operatorname{grad} u(\phi(X,Y)) = u(\phi(X,Y)) = \phi^* u(X,Y)$$

$\hat{\Omega}$:



Exercise 2:

Let $\hat{x} = \phi \hat{x}$,

$$\text{grad}_x u(x) = \left(\frac{\partial}{\partial x_i} u(x) \right)_{i=1}^{N}$$

$$\text{grad}_x u(\phi(\hat{x})) = \left(\frac{\partial}{\partial x_i} u(\phi(\hat{x})) \right)_{i=1}^{N} = \left(\sum_{j=1}^{N} \frac{\partial \hat{x}_j}{\partial x_i} \frac{\partial}{\partial \hat{x}_j} (\phi^* u) \right)$$

$$= (D\phi)^{-T} \text{grad}_{\hat{x}} (\phi^* u).$$

$$\text{grad}_x u(\phi(\hat{x})) \cdot b = f(\phi(\hat{x})).$$

which implies:

$$\left[D\phi^{-T} \right] \text{grad}_{\hat{x}} (\phi^* u) \cdot b = \phi^* f.$$

Exercise 3:

For all $v \in \mathbb{R}^3$ (not depending of x)

$$\left[\int_\Omega \text{curl } q(x)\, dx\right] \cdot v = \int_\Omega \left[\text{curl } q(x) \cdot v\right] dx$$

$$= \int_\Omega \text{div}\left[q(x) \times v\right] dx$$

$$= \int_{\partial\Omega} \left[(q \times v) \cdot n\right] ds$$

$$= \int_{\partial\Omega} \left[(n \times q) \cdot v\right] ds = \left[\int_{\partial\Omega} n \times q \, ds\right] \cdot v$$

$$\Longrightarrow \int_\Omega \text{curl } q(x)\, dx = \int n \times q \; ds$$

Exercise 4.

(i) $-\int_V q(x)\; u^2(x)\; dx = \int_{\partial V} j\cdot n\, ds = \int_V \operatorname{div} j\; dx$, for all volume V

$$\operatorname{div} j = -\; q(x)\; u^2(x)$$

Then, one has:

$$-\operatorname{div} c(x) \operatorname{grad} u(x) \;\left(= \operatorname{div} j(x)\right) = -\,q(x)\; u^2.$$

$$\Rightarrow \begin{cases} -\operatorname{div} c(x) \operatorname{grad} u(x) + q(x)\; u^2(x) = 0 & \text{in } \Omega \\ u = 1 \quad \text{on } \Gamma_D & \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N. \end{cases}$$

(ii) introduce $\tilde{u} = 1$, then $\hat{u} = u - \tilde{u}$

$$-\operatorname{div} c(x) \operatorname{grad} \hat{u} + q(x)\; \hat{u}^2 + 2\hat{u} = -\,q(x) \qquad \text{in } \Omega,$$

$$\hat{u} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial \hat{u}}{\partial n} = 0 \quad \text{on } \Gamma_N,$$

$$\begin{cases} \text{Find } u \in V \text{ such that for all } v \in W \text{ such that:} \\ \int_\Omega c(x)\; \nabla \hat{u} \nabla v + q(x)\; \hat{u}^2(x)\, v(x) + 2\, q(x)\; \hat{u}(x)\, v(x) = -\int_\Omega q(x)\; \hat{u}(x) \end{cases}$$

with

$$V = \left\{ u \in L^4(\Omega) \;/\; Du \in L^2(\Omega), \quad u = 0 \quad \text{on } \Gamma_D \right\}$$

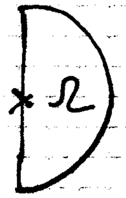
$$W = \left\{ u \in L^2(\Omega) \;/\; Du \in L^4(\Omega), \quad u = 0 \quad \text{on } \Gamma_D \right\}$$

Remark: one can show that $V = W$ if $\Omega \subset \mathbb{R}^d$ with $d = 1, 2, 3$

see [Evans, p 271]

Exercise 5)

We define $\Omega = \{ (x, y) \in \mathbb{R}^2 / \ x > 0 \text{ and } x^2 + y^2 < \frac{1}{2} \}$



We consider $f(r, \varphi) = \log |\log r|$

First, we remark that $f \in H^1(\Omega)$. Indeed

$$\int_\Omega |\nabla f|^2 \, dx = \int_0^{\frac{1}{2}} \int_0^{\pi} \frac{1}{(r |\log r|)^2} \, r \, dr \, d\theta$$

$$= \pi \int_0^{\frac{1}{2}} \frac{1}{r |\log r|^2} \, dr \, d\theta = \left[\frac{-\pi}{\log r} \right]_0^{\frac{1}{2}} = \frac{-\pi}{\log(\frac{1}{2})} = \frac{\pi}{\log 2}$$

$$\int_0^{\frac{1}{2}} \int_0^{\pi} |f(r, \varphi)|^2 \, r \, dr \, d\varphi = \pi \int_0^{\frac{1}{2}} |\log |\log(r)||^2 \, r \, dr < +\infty$$

$$\Rightarrow f \in H^1(\Omega) \Rightarrow f \in H^{\frac{1}{2}}(\partial\Omega)$$

But f is not continuous at $x = 0$ and $y = 0$.