Exercise 1:

(a) Find $u_N \in V_N$ / $A u_N = f_N$ in W_N'

$$\Updownarrow$$

$$\langle A_N u_N ; v_N \rangle_{W_N} = \langle f_N ; v_N \rangle_{W_N} \qquad \forall v_N \in W_N$$

$$\Downarrow \; u_N = \sum u_N^i p_N^i, \text{ pick } v_N = q_N^j$$

$$\sum_{i=1}^{N} u_N^i \langle A_N p_N^i ; q_N^j \rangle_{W_N} = \langle f_N ; q_N^j \rangle_{W_N} \qquad \forall j = 1, \dots, N$$

$$\Updownarrow$$

$$\underline{\underline{A}}_N \, \underline{u}_N = \underline{f}_N$$

$$\Downarrow$$

$$\sum_{i=1}^{N} u_N^i \langle A_N p_N^i ; q_N^j \rangle_{W_N} = \langle f_N ; q_N^j \rangle_{W_N} \qquad \forall j = 1, \dots, N$$

$$\Downarrow$$

$$\langle A_N u_N ; q_N^j \rangle_{W_N} = \langle f_N ; q_N^j \rangle_{W}$$

$$\shortparallel$$

As all $v_N \in W_N$ can be expressed as

$$v_N = \sum_{j=1}^{N} v_N^j q_N^j$$

$$\Downarrow$$

$$\langle A_N u_N ; v_N \rangle_{W_N} = \langle f_N ; v_N \rangle_{W_N} \qquad \forall v_N \in W_N.$$

$$\Downarrow$$

$$A_N u_N = f_N \quad \text{in } W_N'$$

Exercise 1:

b) If $N > P$, then as $Rg\, A_N \subset W_N$ one has

$$\dim(Rg(A_N)) \leqslant P$$

$$\Rightarrow \dim(\mathrm{Ker}(A_N)) = N - \dim(Rg(A_N))$$
$$\geqslant N - P > 0$$

$$\Rightarrow \mathrm{Ker}(A_N) \neq \{0\}$$

A_N is not injective

If $N < P$, then as $\dim(\mathrm{Ker}(A_N)) \geqslant 0$ one has:

$$\dim(Rg(A_N)) \leqslant N < P$$

$$\Rightarrow Rg(A_N) \neq W_N' \Rightarrow A_N \text{ is not surjective}$$

Exercise 2:

Permutation matrix: $\underline{\mathbb{u}}_N = \mathbb{P}\,\mathbb{u}_N$ $\qquad \left(\mathbb{P}^{-1} = \mathbb{P}^T \right)$ $\mathbb{P}$ is real

$$\langle A_N u_N ; v_N \rangle = \langle f_N ; v_N \rangle$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \overline{\mathbb{v}}_N^{j} \, \mathbb{A}_N^{ji} \, \mathbb{u}_N^{i} = \sum_{j=1}^{N} \mathbb{f}_N^{j} \, \mathbb{v}_N^{j}$$

$$\Updownarrow$$

$$\mathbb{v}_N^H \, \mathbb{A}_N \mathbb{u}_N = \mathbb{v}_N^H \, \mathbb{f}_N$$

$$\underline{\mathbb{v}}_N^H \, \underline{\mathbb{A}}_N \, \underline{\mathbb{u}}_N = \underline{\mathbb{v}}_N^H \, \mathbb{f}_N$$

$$\Updownarrow$$

$$\mathbb{v}_N^H \, \mathbb{P}^H \underline{\mathbb{A}}_N \, \mathbb{P}\, \mathbb{u}_N = \mathbb{v}_N^H \, \mathbb{P}^H \, \mathbb{f}_N$$

$$\begin{cases} \mathbb{A}_N = \mathbb{P}^H \, \underline{\mathbb{A}}_N \, \mathbb{P} \\ \mathbb{f}_N = \mathbb{P}^H \, \underline{\mathbb{f}}_N \end{cases}$$

$$\underline{\mathbb{A}}_N = \mathbb{P}\,\mathbb{A}_N \mathbb{P}^H \quad \text{and} \quad \underline{\mathbb{f}}_N = \mathbb{P}\,\mathbb{f}_N .$$

Example of $\mathbb{P}$:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} u_2 \\ u_1 \\ u_3 \end{pmatrix} \qquad \mathbb{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercice 3:

$$\mathbb{A}_N^2 = \left(\langle\langle A_N \alpha_i p_N^i \,;\, \alpha_j p_N^j \rangle\rangle\right)_{i,j=1}^{N} = \left(\alpha_i \alpha_j \, \mathbb{A}_N^{ij}\right)$$

$$\mathbb{f}_N^2 = \left(\langle f_N \,;\, \alpha_j p_N^j \rangle_{W_N}\right)_{j=1}^{N} = \left(\alpha_j \, \mathbb{f}_N^j\right)_{j=1}^{N}$$

Exercise 4:

Let $\{q'^{j}_{N}\}$ be a basis of W'_N:

$$f_N = \sum_{k=1}^{n} \mathbb{f}_N^k \, q'^{k}_N$$

$$A_N u_N = f_N \iff \sum_{i=1}^{N} \langle A_N p_N^i ; q_N^j \rangle = \sum_{k=1}^{n} \mathbb{f}_N^k \langle q'^{k}_N ; q_N^j \rangle$$

To obtain the same equation, on has to ensure for all $j = 1, \dots, N$ and for all $\mathbb{f}$

$$\sum \mathbb{f}_N^k \langle q'^{k}_N ; q_N^j \rangle = \mathbb{f}_N^j$$

$$\left(\langle q'^{k}_N ; q_N^j \rangle_{W_N} \right)_{k,j=1}^{N} = Id$$

We define the linear form $\{q'^{k}_N\}_{k=1}^{N}$ on the basis $\{q_N^j\}_{j=1}^{N}$:

$$q'^{k}_N (q_N^j) = 1 \quad \text{if } k = j \qquad = 0 \quad \text{else}.$$

Exercise 5 :

If $u_N = \sum_{i=1}^{N} < f_N ; q_N^i > p_N^i$

then $\langle A_N u_N * ; q_N^j \rangle = \langle f_N ; q_N^j \rangle$

$$\| $$

$$\sum_{i=1}^{N} \langle f_N ; q_N^i \rangle \langle A_N p_N^i ; q_N^j \rangle$$

As this has to be true for all f_N^i

$$Id = A_N^{ij} \Rightarrow \boxed{A_N^{ij} = Id}$$

and therefore p_N^i is define as the solution of

$$A_N \, p_N^i = q'^{\,i}_N$$

where $q'^{\,i}_N$ is the form defined by the basis $\mathcal{B}\mathcal{W}$ via

$$q'^{\,i}_N (q_N^j) = 1 \text{ if } i=j, \quad =0 \text{ else.}$$