Exercise 1:

We recall that

$$\int_{-1}^{1} L_n(x) L_p(x)\, dx = 0 \qquad \text{for } n \neq p$$

$$\int_{-1}^{1} L_n^2(x)\, dx = \frac{1}{n+\frac{1}{2}}$$

For $n \geqslant p$

$$\int L_n'(x)\, L_p'(x)\, dx = -\underbrace{\int L_n(x)\, L_p''(x)\, dx}_{=0} + \Big[L_n(x)\, L_p'(x) \Big]_{x=-1}^{1}$$

$$= L_n(1)\, L_p'(1) - L_n(-1)\, L_p'(-1)$$

$$= 1 \cdot \frac{1}{2} p(p+1) - (-1)^n (-1)^{p+1} \frac{1}{2} p(p+1)$$

$$= \frac{1}{2} p(p+1) \left[1 + (-1)^{n+p} \right]$$

$$= \begin{cases} p(p+1) & \text{if } n+p \text{ even} \\ 0 & \text{if } n+p \text{ odd} \end{cases}$$

Consider $u \in P_N(\Lambda)$

$$u(x) = \sum_{n=0}^{N} \alpha_n L_n(x)$$

$$\|u\|^2_{L^2(\Lambda)} = \sum_{n=0}^{N} \sum_{p=0}^{N} \alpha_n \alpha_p \int_{-1}^{1} L_n(x) L_p(x)\, dx$$

$$= \sum_{n=0}^{N} \frac{\alpha_n^2}{n+\frac{1}{2}}$$

$$|u|^2_{H^1(\Lambda)} = \sum_{n=0}^{N} \sum_{p=0}^{N} \alpha_n \alpha_p \int_{-1}^{1} L'_n(x) L'_p(x)\, dx$$

$$= \sum_{n=0}^{N} \alpha_n^2 \int_{-1}^{1} \left[L'_n(x)\right]^2 dx + 2 \sum_{n,p / n>p} \alpha_n \alpha_p \int_{-1}^{1} L'_n(x) L'_p(x)\, dx$$

$$= \sum_{n=0}^{N} \alpha_n^2 n(n+1) + \sum_{n=0}^{N} \sum_{\substack{p<n \\ n+p \text{ even}}} \alpha_n \alpha_p \, 2p(p+1)$$

We choose $\alpha_n = \sqrt{n+\frac{1}{2}}$

$$\|u\|^2_{L^2(\Lambda)} = N$$

$$|u|^2_{H^1(\Lambda)} = \sum_{n=0}^{N} \left(n+\tfrac{1}{2}\right) n(n+1) + \sum_{n=0}^{N} \sum_{\substack{p<n \\ p+n \text{ even}}} (n+\tfrac{1}{2})^{1/2} \underbrace{(p+\tfrac{1}{2})^{1/2}\, 2p(p+1)}$$

$$|u|^2_{H^1(\Lambda)} \geq C_1 N^4 + C_1 \sum_{n=0}^{N} \underbrace{n^{1/2} \; n^{7/2}}_{= n^4} \quad \text{for } n \text{ large}$$

$$|u|^2_{H^1(\Lambda)} \geq C N^5$$

Hence, $\exists u \in P_N$ such that

$$|u|_{H^1(\Lambda)} \geq C N^2 \|u\|_{L^2(\Lambda)}$$