Exercie 1:

(i) * As a and b are continuous bilinear forms

$a_\varepsilon = a + \varepsilon b$ is a bilinear continuous form.

* V-ellipticity:

$$|a_\varepsilon(u,u)| \geqslant |a(u,u)| - |\varepsilon|\,|b(u,u)|$$
$$\geqslant \alpha \|u\|^2 - |\varepsilon|\,\|b\|\,\|u\|^2 = (\alpha - |\varepsilon|\,\|b\|)\,\|u\|^2$$

$\alpha - |\varepsilon|\,\|b\| > 0$ iff $|\varepsilon| < \frac{\alpha}{\|b\|} =: \varepsilon_0$

* $f \in W'$

$\Longrightarrow$ (P) has a unique solution for $|\varepsilon| < \varepsilon_0$.

Exercie 1:

(ii)

$$\begin{cases} a(u_\varepsilon, v) + \varepsilon\, b(u_\varepsilon, v) = f(v) & \forall v \in V \\ a(u_0, v) \qquad\qquad\qquad = f(v) & \forall v \in V \end{cases}$$

Hence, one has:

$$a(u_\varepsilon - u_0) + \varepsilon\, b(u_\varepsilon - u_0, v) = \varepsilon b(u_0, v)$$

Take $v = u_\varepsilon - u_0$

$$a(u_\varepsilon - u_0; u_\varepsilon - u_0) + \varepsilon\, b(u_\varepsilon - u_0; u_\varepsilon - u_0) = \varepsilon b(u_0, u_\varepsilon - u_0)$$

$$\left(\alpha - |\varepsilon|\,\|b\|\right) \|u_\varepsilon - u_0\|^2 \leq |\varepsilon| \|b\|\,\|u_0\|\,\|u_\varepsilon - u_0\|$$

$$\|u_\varepsilon - u_0\| \leq \frac{|\varepsilon|}{\varepsilon_0 - |\varepsilon|} \|u_0\| \qquad (\alpha)$$

As $a(u_0; v) = f(v) \quad \forall v \in V$

Take $v = u_0$: $\alpha \|u_0\|^2 \leq a(u_0, u_0) = f(u_0) \leq \|f\|\,\|u_0\|$

$$\Longrightarrow \|u_0\| \leq \frac{\|f\|}{\|\alpha\|} \qquad (\beta)$$

$$(\alpha), (\beta) \Longrightarrow \|u_\varepsilon - u_0\| \leq \frac{|\varepsilon|}{\varepsilon_0 - |\varepsilon|} \|u_0\| \leq \frac{|\varepsilon|}{\varepsilon_0 - |\varepsilon|} \frac{\|f\|}{\alpha}$$

(iii) When $\varepsilon \to 0$, we get $\|u_\varepsilon - u_0\|_V \to 0$ and therefore

$$u_\varepsilon \to u_0 \quad \text{in } V.$$

Ex 2:

(i) a is a continuous bilinear form in V (α)

$\Rightarrow a$ is a continuous bilinear form in V_n (α')

By hypothesis,

a satisfies inf-sup condition in V (β)

a ——— in V_n (β')

$\forall w \in V \subset W$, one has

$$|f(v)| \leqslant \|f\|_{W'} \|v\|_W \leqslant C_1 \|f\|_{W'} \|v\|_V \quad (\delta)$$

$\Rightarrow f$ is a linear continuous form <u>in V</u> (γ)

an in V_n (γ')

$$(\alpha), (\beta), (\gamma) \xRightarrow{\text{inf-sup Thm}} (P) \text{ has a unique solution}$$

$$\text{and } \|u\|_V \leqslant \frac{1}{\inf\limits_{v \in V \setminus \{0\}} \sup\limits_{w \in V \setminus \{0\}} \frac{|a(v,w)|}{\|v\|_V \|w\|_V}} \|f\|_{V'}$$

Due to (δ), $\|f\|_{V'} \leqslant C_1 \|f\|_{W'}$.

$$\Rightarrow \|u\|_V \leqslant \frac{C_1}{\alpha} \|f\|_{W'}.$$

$$(\alpha'), (\beta'), (\gamma') \xRightarrow{\text{inf-sup thm}} (P_n) \text{ has a unique solution}$$

$$\text{and } \|u_n\|_V \leqslant \frac{C_1}{\alpha} \|f\|_{W'}$$

Exercie 2:

(ii) Due to abstract a priori estimate theorem, one has

$$\|u-u_n\|_V \leqslant \left(1+\frac{\|a\|}{\alpha}\right) \inf_{v_n\in V_n} \left(\|u-v_n\|_V\right)$$

$$\leqslant \left(1+\frac{\|a\|}{\alpha}\right) \frac{c_3}{n} \|u\|$$

When $n \longrightarrow +\infty$, $\|u-u_n\| \longrightarrow 0$

$$\Longrightarrow \boxed{\lim_{n\to+\infty} u_n = u}$$

Exercise 3:

V_n is finite dimensional linear space
$\Longrightarrow V_n$ is closed.

Therefore as $V_n \subset V$, V_n admits an orthogonal $V_n^{\perp}$ $\left(V = V_n \overset{\perp}{\oplus} V_n^{\perp}\right)$

We act by contradiction:

if for all $v \in V \inf_{v_n \in V_n} \| v - v_n \| \leqslant \alpha \|v\|$ $(\alpha < 1)$

This is also true for $v \in V_n^{\perp} \setminus \{0\}$

but as $v \perp v_n$ one has

$$\|v - v_n\|^2 = \|v\|^2 + \|v_n\|^2 \Longrightarrow \|v - v_n\| \geqslant \|v\|$$

Hence

$$\|v\| \leqslant \inf \|v - v_n\| \leqslant \alpha \|v\| \Rightarrow \underbrace{(1-\alpha)}_{>0} \|v\| \leqslant 0$$

$$\Rightarrow v = 0$$

impossible

Exercice 4:

uniqueness:

if $(u_1^1 ; u_2^1)$ are two solutions, then
$(u_1^2 ; u_2^2)$

then $(u_1 = u_1^2 - u_1^1 ; u_2 = u_2^2 - u_2^2)$ is solution of

$$a(u_1, v_1) + a(u_2, v_1) + a(u_2 ; v_2) = 0 \quad \forall v_1 ; v_2 \in V$$

Choose $v_1 = 0, \quad v_2 = u_2)$

$$a(u_2, u_2) = 0 \underset{V\text{-ellipticity}}{\Longrightarrow} \alpha \|u_2\|^2 \leqslant 0 \Rightarrow u_2 = 0$$

Hence, one has:

$$a(u_1, v_1) = 0 \quad \forall v_1 \in V$$

Pick $v_1 = u_1$ $\quad a(u_1 ; u_1) = 0 \underset{V\text{ellipticity}}{\Longrightarrow} \alpha \|u_1\|^2 \leqslant 0 \Rightarrow u_1 = 0$

$$\left.\begin{matrix} u_1 = 0 \\ u_2 = 0 \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} u_1^1 = u_1^2 \\ u_2^1 = u_2^2 \end{matrix}\right. \qquad \text{Unicity !}$$

Exercice 4

Existence:

If u_1, u_2 are solutions then

u_2 is the unique solution of the problem (Pick $v_1=0$

(α) $a(u_2, v_2) = f(v_2)$ $\forall v_2 \in V$.
- a is V-elliptic
- a is C^0
- $f \in V'$

u_2 is now a <u>data</u>:

u_1 is the unique solution of

(β) $a(u_1; v_1) = -a(u_2, v_1)$ $\forall v_1 \in V$

- a is V-elliptic, a is C^0
- and $v_2 \mapsto a(u_1, v_2)$ is continuous as

$$|a(u_1, v_1)| \leqslant (\|a\| \, \|u_1\|) \|v_1\|$$

Reciprocally if u_1, u_2 solve (i) and (ii) then they are solutions.