Exercice 1 :

$$\nabla \cdot (f \vec{v}) = \operatorname{div}(f \vec{v})$$

$$= \partial_x (f v_x) + \partial_y (f v_y) + \partial_z (f v_z)$$

$$= (\partial_x f) v_x + (\partial_y f) v_y + (\partial_z f) v_z$$

$$+ f \partial_x v_x + f \partial_y v_y + f \partial_z v_z$$

$$= \vec{\nabla} f \cdot \vec{v} + f \operatorname{div} \vec{v}.$$

Exercice 2 :

$$\operatorname{div} \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$J_{(r,\theta,z) \to (x,y,z)} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_{(x,y,z) \to (r,\theta,z)} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}$$

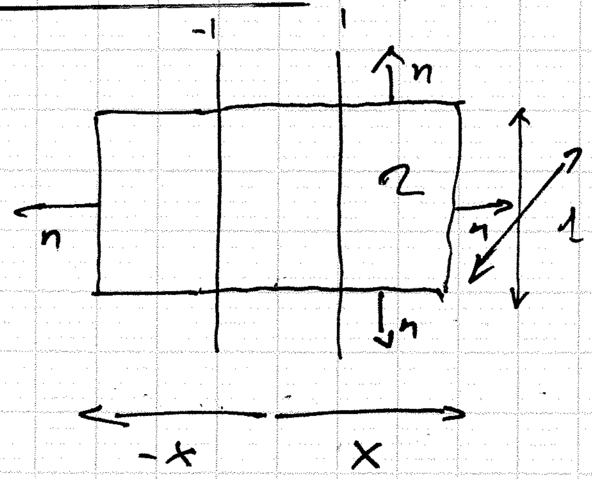
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$$v_x = v_r \cos\theta - v_\theta \sin\theta$$
$$v_y = v_r \sin\theta + v_\theta \cos\theta$$
$$v_z = v_z$$

$$\begin{aligned}
\operatorname{div} \vec{v} = {} & \cos^2\theta \frac{\partial}{\partial r} v_r - \cos\theta \sin\theta \frac{\partial v_\theta}{\partial r} \\
& + \sin^2\theta \frac{\partial}{\partial r} v_r + \cos\theta \sin\theta \frac{\partial v_\theta}{\partial r} \\
& - \frac{1}{r} \sin\theta \frac{\partial}{\partial \theta}\left(v_r \cos\theta - v_\theta \sin\theta\right) \\
& + \frac{1}{r} \cos\theta \frac{\partial}{\partial \theta}\left(v_r \sin\theta + v_\theta \cos\theta\right) \\
& + \frac{\partial}{\partial z} v_z \\
= {} & \frac{\partial}{\partial r} v_r + \cancel{\text{[illegible]}} \\
& + \frac{1}{r}\left(\sin^2\theta + \cos^2\theta\right) v_r + 0 \\
& + \frac{1}{r}\left(\sin^2\theta + \cos^2\theta\right) \frac{\partial v_\theta}{\partial \theta} \\
& + \frac{\partial v_z}{\partial z}
\end{aligned}$$

$$\operatorname{div} \vec{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \partial_\theta v_\theta + \frac{\partial}{\partial z} v_z .$$

Exercice 3.



On applique le théorème de la divergence :

$$\int_{\Omega} \operatorname{div} \vec{E}(\vec{x})\, dx = \int_{\partial\Omega} \vec{E}(\vec{x}).\vec{n}\, ds$$

$$\int_{\Omega} e(\vec{x})\, dx = 2 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} E_x(x)\, dy\, dz = 2E_x(x)$$

si $x<1$ $\displaystyle\int_{\Omega} e(\vec{x})\, dx = 2\int_0^x \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (1-x)\, dz\, dy\, dx$

$$= 2\left(x - \frac{x^2}{2}\right) \Longrightarrow E_x(x) = x - \frac{x^2}{2}$$

si $x>1$ $\displaystyle\int_{\Omega} e(\vec{x})\, dx = 2\int_0^1 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (1-x)\, dz\, dy\, dz$

$$= 1 \Longrightarrow E_x(x) = \frac{1}{2}.$$

Exercice 4 :

théorème de la divergence :

$$\int_{B_r} \operatorname{div} \vec{E}(\vec{x})\, dx = \int_{S_r} \vec{E}\cdot\vec{n}\, dS = 4\pi r^2 E_r(r)$$

si $r < R$

$$\int_0^r \int_0^{\pi} \int_0^{2\pi} \left(1 - \frac{r^2}{R^2}\right) r^2 \sin\theta \, d\varphi\, d\theta\, dr$$

$$= \int_0^r \left(1 - \frac{r^2}{R^2}\right) 4\pi r^2 dr = 4\pi\left(\frac{r^3}{3} - \frac{r^3}{5}\cdot\frac{r^2}{R^2}\right)$$

$$= 4\pi r^3 \left(\frac{1}{3} - \frac{r^2}{5R^2}\right)$$

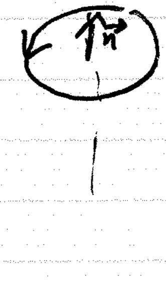
$$E_r(r) = r\left(\frac{1}{3} - \frac{r^2}{5R^2}\right)$$

si $r > R$

$$\int_{B_r} \operatorname{div} \vec{E}(\vec{x})\, dx = 4\pi R^3 \cdot \frac{2}{15}$$

$$E_r(r) = \frac{2}{15}\frac{R^3}{r^2}$$

Exercice 5 :



$$\int_0^r \int_0^{2\pi} \operatorname{rot} B \cdot n \; r\, dr\, d\theta = \int \vec{B}\cdot\vec{dl}$$

$$\int_0^r \int_0^{2\pi} r^2 dr\, d\theta = 2\pi r B_\theta(r)$$

$$2\pi \frac{r^3}{3} = 2\pi r B_\theta(r)$$

$$B_\theta(r) = \frac{r^2}{3}$$