Exercice 1

- $x^2 + y^2 = R^2 \sin^2\theta$ (coordonnée sphérique)

- $x = R \sin\theta \cos\varphi$, $y = R \sin\theta \sin\varphi$, $z = R\cos\theta$; $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\frac{\partial \vec{x}}{\partial \theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = R\,\vec{e_\theta}$$

$$\frac{\partial \vec{x}}{\partial \varphi} = \begin{pmatrix} R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \end{pmatrix} = R\sin\theta\,\vec{e_\varphi}$$

$$\frac{\partial \vec{x}}{\partial \theta} \wedge \frac{\partial \vec{x}}{\partial \varphi} = R^2 \sin\theta\,\vec{e_r}$$

$$\left\| \frac{\partial \vec{x}}{\partial \theta} \wedge \frac{\partial \vec{x}}{\partial \varphi} \right\|_2 = R^2 \sin\theta$$

$$\int_{S_R} x^2 + y^2 \, dS = \int_0^\pi \left(\int_0^{2\pi} R^4 \sin^3\theta \, d\varphi \right) d\theta$$

$$= 2\pi R^4 \int_0^\pi \sin^3\theta \, d\theta = 2\pi R^4 \int_0^\pi \sin\theta - \sin\theta\cos^2\theta \, d$$

$$= 2\pi R^4 \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_0^\pi = 2\pi R^4 \cdot \frac{4}{3} = \frac{8}{3}\pi R^4$$

Exercice 2

partie latérale

$$\vec{x} = R\cos\theta \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$y = R\sin\theta$$
$$z = z$$

$$\frac{d\vec{x}}{d\theta} = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix} = R\vec{e_\theta}$$

$$\frac{d\vec{x}}{dz} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e_z}$$

$$\frac{d\vec{x}}{d\theta} \wedge \frac{d\vec{x}}{dz} = R\vec{e_r}$$

$$\left\| \frac{d\vec{x}}{d\theta} \wedge \frac{d\vec{x}}{dz} \right\|_2 = R$$

Partie haute et basse

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 0 \text{ ou } H$$

$$\frac{d\vec{x}}{dr} = \vec{e_r} \text{ et } \frac{d\vec{x}}{d\theta} = r\vec{e_\theta} \qquad \frac{d\vec{x}}{dr} \wedge \frac{d\vec{x}}{d\theta} = r\vec{e_z}$$

$$\left\| \frac{d\vec{x}}{dr} \wedge \frac{d\vec{x}}{d\theta} \right\|_2 = r$$

$$S = \int_0^{2\pi}\int_0^H R\,d\theta\,dz + 2\int_0^R\int_0^{2\pi} r\,dr\,d\theta = 2\pi RH + 2\pi R^2$$

Exercice 3 :

$$\begin{cases} x = z \cos\theta \\ y = z \sin\theta \\ z = z \end{cases}$$

$$\frac{\partial \vec{x}}{\partial \theta} = \begin{pmatrix} -z \sin\theta \\ z \cos\theta \\ 0 \end{pmatrix} = z\,\vec{e}_\theta \qquad \frac{\partial \vec{x}}{\partial z} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} = \vec{e}_r + \vec{e}_z$$

$$\frac{\partial \vec{x}}{\partial \theta} \wedge \frac{\partial \vec{x}}{\partial z} = z\;\vec{e}_\theta \wedge \left(\vec{e}_r + \vec{e}_z\right)$$

$$= +z\left(-\vec{e}_z + \vec{e}_r\right)$$

$$\left\| \frac{\partial \vec{x}}{\partial \theta} \wedge \frac{\partial \vec{x}}{\partial z} \right\|_2 = \sqrt{2} \cdot z .$$

$$S = \int 1 \, dS = \int_0^H \left(\int_0^{2\pi} \sqrt{2}\, z \, d\theta \right) dz$$

$$= 2\pi \sqrt{2} \int_0^H \frac{z^2}{2} dz$$

$$= \pi \sqrt{2}\; H$$

Exercice 4 :

$$x = \cos\theta \qquad \frac{d\vec{x}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 1 \end{pmatrix} = \vec{e}_\theta + \vec{e}_z$$
$$y = \sin\theta$$
$$z = \theta$$

$$\left\| \frac{d\vec{x}}{d\theta} \right\|_2 = \sqrt{2}$$

$$xz = \theta\cos\theta$$

$$\int_{\mathcal{C}} xz \, dl = \int_0^{2\pi} \theta\cos\theta \sqrt{2}\, d\theta$$

$$= \Big[\theta \sin\theta\Big]_{\theta=0}^{\theta=2\pi} \sqrt{2} - \int_0^{2\pi} \sin(\theta) \sqrt{2}\, d\theta$$

$$= 0.$$

Exercice 5 :

$\vec{x}$ un point du triangle

$$\alpha + \beta = 1, \qquad 0 \leq \alpha \leq 1 \text{ et } 0 \leq \beta \leq 1$$

$$x = (1-\alpha-\beta)\,0 + \alpha + \beta.0$$
$$y = (1-\alpha-\beta)\,0 + \alpha + \beta.$$
$$z = (1-\alpha-\beta)\,0 + \alpha.0 + \beta$$

$$x = \alpha, \qquad y = \alpha + \beta \quad \text{et } z = \beta$$

$$\frac{\partial \vec{x}}{\partial \alpha} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial \beta} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\frac{\partial \vec{x}}{\partial \alpha} \wedge \frac{\partial \vec{x}}{\partial \beta} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\left\| \frac{\partial \vec{x}}{\partial \alpha} \wedge \frac{\partial \vec{x}}{\partial \beta} \right\|_2 = \sqrt{3}$$

$$\int_S x \, dS = \int_0^1 \left(\int_0^{1-\alpha} \sqrt{3}\,\alpha \, d\beta \right) d\alpha$$

$$= \int_0^1 \sqrt{3}\,\alpha\,(1-\alpha)\, d\alpha$$

$$= \sqrt{3}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$

Exercice 6 :

$$\begin{aligned} x &= t \\ y &= t^2 \end{aligned} \qquad \frac{d\vec{x}}{dt} = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \qquad \left\| \frac{dx}{dt} \right\|_2 = \sqrt{1+4t^2}$$

$$\frac{1}{\sqrt{1+4y}} \exp(-\sqrt{y}) = \frac{1}{\sqrt{1+4t^2}} \exp - |t|$$

$$\int_\varphi \frac{1}{\sqrt{1+4y}} \exp(-\sqrt{y})\, dl = \int_{-\infty}^{+\infty} \exp - |t| \, dt = 2$$