Mathematical Models and Methods in Applied Sciences
Vol. 19, No. 10 (2009) 1853–1882

Doi: 10.1142/S021820250900398X

INTERACTIONS BETWEEN MODERATELY CLOSE INCLUSIONS FOR THE LAPLACE EQUATION

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Received 31 March 2008
Revised 8 December 2008
Communicated by S. Müller

The presence of small inclusions modifies the solution of the Laplace equation posed in a reference domain Ω_0. This question has been studied extensively for a single inclusion or well-separated inclusions. In two-dimensional situations, we investigate the case where the distance between the holes tends to zero but remains large with respect to their characteristic size. We first consider two perfectly insulated inclusions. In this configuration we give a complete multiscale asymptotic expansion of the solution to the Laplace equation. We also address the situation of a single inclusion close to a singular perturbation of the boundary $\partial\Omega_0$. We also present numerical experiments implementing a multiscale superposition method based on our first order expansion.

Keywords: Multiscale asymptotic expansion; Laplace equation; singular shape perturbation; numerical simulations.

AMS Subject Classification: 35C20, 35J25, 65N30

1. Introduction

The presence of small inclusions or surface defects alters the solution of the Laplace equation posed in a reference domain Ω_0. If the characteristic size of the perturbation is small, one can expect the solution of the problem posed on the perturbed geometry to be close to the solution of the reference shape. An asymptotic expansion with respect to that small parameter — the characteristic size of the perturbation — can then be performed.

The case of a single inclusion ω, centered at the origin 0 being either in Ω_0 or on $\partial\Omega_0$, has been deeply studied, see Refs. 12, 9, 10, 15, 6, 7 and 1. The techniques rely on the notion of *profile*, a normalized solution of the Laplace equation in the exterior domain obtained by *blow-up* of the perturbation, see (1.2). It is used in a fast variable to describe the local behavior of the solution in the perturbed domain. Convergence of the asymptotic expansion is obtained thanks to the decay of the *profile* at infinity. For example, if we impose Neumann boundary conditions on the inclusion and Dirichlet on $\partial\Omega_0$, the expansion takes the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon V_0\Big(\frac{x}{\varepsilon}\Big) + r_\varepsilon^1(x), \quad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2), \tag{1.1}$$

where

- u_0 is the solution of the Laplace–Dirichlet problem in Ω_0: $u_0 \in H_0^1(\Omega_0)$, $-\Delta u_0 = f$,
- V_0 is a *profile* satisfying

$$\begin{cases} -\Delta V_0 = 0 & \text{in } \mathbb{R}^2\setminus\overline{\omega}, \\ \partial_{\mathbf{n}} V_0 = -\nabla u_0(0)\cdot\mathbf{n} & \text{on } \partial\omega, \\ V_0 \to 0 & \text{at infinity,} \end{cases} \tag{1.2}$$

where $\mathbf{n}$ denotes the unit normal vector pointing into ω.

We present, in this work all the proofs of the results announced in Ref. 3. We consider, in two-dimensional situations, the case of two singular perturbations. Let Ω_0, ω^-, and ω^+ be three bounded domains of $\mathbb{R}^2$, each containing the origin 0. For $\varepsilon > 0$, small enough, we define the perturbed domain Ω_ε as

$$\Omega_\varepsilon = \Omega_0\setminus\overline{(\omega_\varepsilon^- \cup \omega_\varepsilon^+)}, \quad \text{with } \omega_\varepsilon^\pm = x_\varepsilon^\pm + \varepsilon\omega^\pm, \tag{1.3}$$

where $x_\varepsilon^\pm = \pm\eta_\varepsilon\mathbf{d}$ with a given unitary vector $\mathbf{d}$, and a real number η_ε. Shortly, Ω_ε consists of Ω_0 from which two ε-inclusions at distance $2\eta_\varepsilon$ have been removed, cf. Fig. 1(a).

We aim at building an asymptotic expansion of the solution u_ε of the Laplace problem in Ω_ε

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma = \partial\Omega_0, \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon^\pm, \end{cases} \tag{1.4}$$

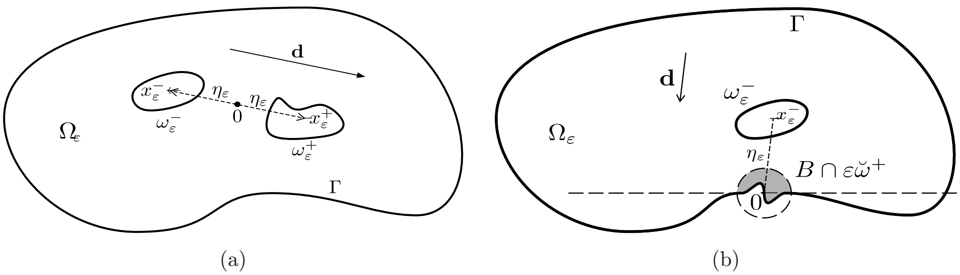


Fig. 1. Geometrical settings for perturbed domains. (a) Two interior inclusions of size ε, at distance $2\eta_\varepsilon$, (b) Boundary perturbation.

for some L^2 datum f whose support does not contain the origin 0. This is only a technical restriction. In the general case, the asymptotic expansion contains extra terms due to the expansion of f near the origin 0, see Ref. 4. We restrict ourselves to homogeneous Neumann boundary conditions on $\partial\omega_\varepsilon^\pm$, although generalizations to other conditions are possible. Besides, one of the inclusions may be localized at the boundary Γ of Ω_0 (or even simply be removed, the remaining inclusion moving towards the external boundary), see Fig. 1(b). Note that the origin now lies on $\partial\Omega_0$.

The results obtained previously for a single perturbation easily extend to the case of two (or finitely many) inclusions within two situations:

1. *Inclusions at distance* $\mathcal{O}(1)$. It corresponds to $\eta_\varepsilon = \eta$ independent of ε. In this case considered in Ref. 13, Sec. 5.3, the centers $x^\pm$ are independent of ε. The decaying profiles $V_0^\pm$ are harmonic in $\mathbb{R}^2\setminus\overline{\omega^\pm}$ and satisfy the boundary conditions

$$\partial_{\mathbf{n}} V_0^\pm = -\nabla u_0(x^\pm)\cdot\mathbf{n} \quad \text{on } \partial\omega^\pm.$$

At the first order, the holes do not interact with each other, their contributions are merely superposed

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^+\left(\frac{x-x^+}{\varepsilon}\right) + V_0^-\left(\frac{x-x^-}{\varepsilon}\right)\right] + r_\varepsilon^1(x), \qquad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2). \tag{1.5}$$

2. *Inclusions at distance* $\mathcal{O}(\varepsilon)$. It corresponds to $\eta_\varepsilon = c\,\varepsilon$ with a constant $c \in \mathbb{R}$. Here the two inclusions constitute a unique pattern at the scale ε. This case is actually handled as a single inclusion $\omega = \omega^+ \cup \omega^-$, self-similar with respect to the origin 0. The expansion reads

$$u_\varepsilon(x) = u_0(x) + \varepsilon W_0\left(\frac{x}{\varepsilon}\right) + r_\varepsilon^1(x), \quad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2), \tag{1.6}$$

where the profile W_0 is associated with the whole pattern ω.

These two situations show radically different behaviors: no interaction and full interaction. Here we focus on the intermediate cases, where the inclusions are *moderately close*, i.e.

$$\eta_\varepsilon \to 0 \quad \text{and} \quad \eta_\varepsilon/\varepsilon \to +\infty \quad (\text{as } \varepsilon \to 0). \tag{1.7}$$

One can expect to have a weak interaction between the two inclusions. To quantify this effect, we specify the range η_ε as $\eta_\varepsilon = \varepsilon^\alpha$ with $\alpha \in (0, 1)$. Other scales could be considered as well but ε^α is rather natural and will lead to interesting regimes of interaction. The limit case $\alpha = 0$ corresponds to inclusions at distance $\mathcal{O}(1)$ while the other limit $\alpha = 1$ corresponds to inclusions at distance $\mathcal{O}(\varepsilon)$. Let us mention that a three-scale problem has been treated in Ref. 13, Sec. 5.4, Example 5.4.2. It consists in a bump at scale $\varepsilon^{1+\kappa}$ on a ε-boundary singular perturbation of a smooth domain. Some techniques involved are close to ours and the geometrical setting is different.

This work is organized as follows. In Sec. 2, we precise the geometrical setting we shall work within and state the multiscale asymptotic expansions. In Sec. 3, we gathered all the preliminary results needed to construct and justify the expansions of the solutions of the considered boundary value problems. Section 4 is devoted to the proofs of Theorems 2.1 and 2.2. Finally in Sec. 5, we show numerical results obtained with the first order approximation, confirming our theoretical results. We also discuss the limitation in ε of the asymptotic regime as well as alternative correction methods.

2. Multiscale Asymptotic Expansions

We now consider the situation of Fig. 1, where the distance between the two inclusions equals ε^α with $\alpha \in (0, 1)$, and we focus on the following two-dimensional problems which cover the main difficulties and techniques: $u_\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies the Laplace equation $-\Delta u_\varepsilon = f$ with various boundary conditions, see Fig. 1:

(a) *two Neumann inclusions*: $u_\varepsilon = 0$ on Γ and $\partial_{\mathbf{n}} u_\varepsilon = 0$ on $\partial\omega_\varepsilon^- \cup \partial\omega_\varepsilon^+$,
(b) *a Neumann inclusion and a Dirichlet boundary perturbation*[a]: $\partial_{\mathbf{n}} u_\varepsilon = 0$ on $\partial\omega_\varepsilon^-$, $u_\varepsilon = 0$ elsewhere.

We start with giving a brief description of the first terms in the expansions. Theorems 2.1 and 2.2 state the complete asymptotics with optimal remainder estimates.

Case (a). For two Neumann inclusions, centered respectively in x_ε^- and x_ε^+ (separated by a distance $2\varepsilon^\alpha$), the first correctors involve the profiles $V_0^\pm$ as introduced in (1.2)

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x), \tag{2.1}$$

with $\|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)}$

$$= \mathcal{O}(\varepsilon^{\min(1+\alpha, 3-2\alpha)}).$$

[a] In this case, the definition of the perturbed domain Ω_ε is slightly different, see Ref. 7 or later.

The profiles satisfy $\|V_0^{\pm}(\frac{\cdot\ - x_\varepsilon^{\pm}}{\varepsilon})\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(1)$ and only depend on the shape of $\omega^{\pm}$ and on the gradient of the limit term at the origin $\nabla u_0(0)$. We emphasize that the origins $x_\varepsilon^{\pm}$ of the profiles do vary with ε, unlike $x^{\pm}$ in Eq. (1.5) and 0 in (1.6). Moreover, the remainder is of order ε as $\alpha \to 0$ or $\alpha \to 1$ because of the inadequacy of the profiles with the geometry.

We may understand expansion (2.1) in the following way: the main contribution of the two inclusions is merely the superposition of their individual effects. The remainder r_ε^1 contains information about higher-order influence. It is interesting to describe further the structure of this remainder:

- for $\alpha < 2/3$, the inclusions are relatively far away from each other. The leading term in r_ε^1 is $\mathcal{O}(\varepsilon^{1+\alpha})$ and arises from the Taylor expansion of u_0 at the origin 0;
- for $2/3 < \alpha < 1$, the inclusions are closer. The remainder r_ε^1 is $\mathcal{O}(\varepsilon^{3-2\alpha})$ and mainly consists in the *interaction* between the profiles V_0^- and V_0^+;
- for $\alpha = 2/3$, the two contributions are equally balanced.

Theorem 2.1. *The solution u_ε of problem (1.4) admits the expansion at order N*

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] \\ &\quad + \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right]\right) \\ &\quad + r_\varepsilon^N(x), \end{aligned}$$

with

$$\mathcal{K}_N = \left\{(p,q) \in \mathbb{Z}^2 \mid p \geq 0,\ q \geq -\frac{3}{2}p + 1,\ q \geq -p \text{ and } p + \alpha q \leq N\right\},$$

(*see Fig.* 2),

$$\|r_\varepsilon^N\|_{H^1(\Omega_\varepsilon)} = o(\varepsilon^N),$$

and the terms $v_{p+\alpha q}$, $V_{p+\alpha q}^{\pm}$ are built inductively, see Remark 4.1.

Case (b). This situation requires a slightly different definition of the geometry because there is no inclusion between Ω_ε and Ω_0. The origin 0 is assumed to be on the boundary Γ of Ω_0, and Γ to coincide with a straight line in a neighborhood of 0 (the curved case is not a mere adaptation since the perturbation is not self-similar any more after straightening, see Ref. 7). The perturbed domain Ω_ε is defined as

$$\Omega_\varepsilon = [\Omega_0 \setminus (\overline{\omega_\varepsilon^-} \cup B)] \cup (B \cap \varepsilon\breve{\omega}^+), \tag{2.2}$$

where B is a small (but fixed with respect to ε) ball centered in 0 and $\breve{\omega}^+$ is a perturbed upper half plane. Precisely, $\partial\breve{\omega}^+$ is composed of three parts: two horizontal

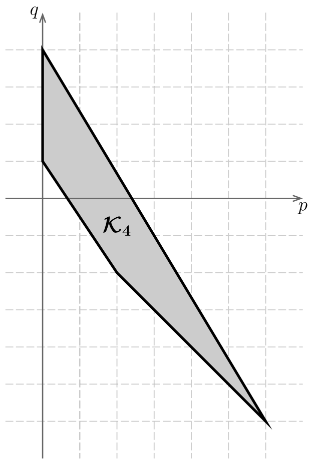


Fig. 2. The set of indices $\mathcal{K}_4$ for $\alpha = \frac{3}{5}$.

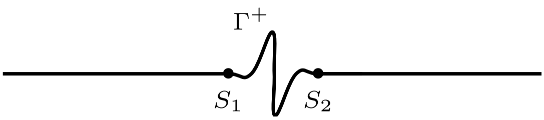


Fig. 3. Geometrical setting for perturbed upper half plane.

straight half lines rising from S_1 and S_2 (two points on the x-axis) and a Lipschitz and rectifiable curve Γ^+ connecting S_1 to S_2 (see Fig. 3).

As explained in Refs. 7 and 17, the inclusion $\Omega_\varepsilon \subset \Omega_0$ may not be satisfied: a cutoff function has to be introduced to define a counterpart for u_0 on Ω_ε. Precisely, the asymptotic expansion takes the form

$$u_\varepsilon(x) = \zeta\left(\left|\frac{x}{\varepsilon}\right|\right)u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\left(\frac{x}{\varepsilon}\right)\right] + r_\varepsilon^1(x), \quad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = o(\varepsilon), \tag{2.3}$$

where $\zeta(r)$ vanishes for $r < r_\bullet$ and $\zeta(r) = 1$ for $r > r^\bullet$, and $\chi(r) = 1$ for $r < r_*$ and $\chi(r) = 0$ for $r > r^*$. The remarks about the interaction between the two perturbations in case (a) still hold.

Theorem 2.2. *The solution u_ε of*

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \backslash \partial\omega_\varepsilon^-, \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon^-, \end{cases} \tag{2.4}$$

admits the expansion at order N

$$u_\varepsilon(x) = \zeta\Big(\Big|\frac{x}{\varepsilon}\Big|\Big)u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\Big(\frac{x}{\varepsilon}\Big)\right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(\zeta\Big(\Big|\frac{x}{\varepsilon}\Big|\Big)v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_{p+\alpha q}^+\Big(\frac{x}{\varepsilon}\Big)\right]\right)$$

$$+ r_\varepsilon^N(x),$$

with $\mathcal{K}_N$ defined in Theorem 2.1 and

$$\|r_\varepsilon^N\|_{H^1(\Omega_\varepsilon)} = o(\varepsilon^N).$$

Remark 2.1. The question of adapting these results to alternative situations and, in particular, to the n-dimensional case ($n \geq 3$) is very natural. The key for proving Theorems 2.1 and 2.2 (see Sec. 4) is the behavior of the profiles at infinity: do they decrease with respect to the distance to the origin or not? In dimension n, the profiles decrease at infinity (see Ref. 2) and the methods used in the present paper can be applied. However, all proofs (from the scaling of Sobolev norms to the behavior of profiles and the construction of the expansion) have to be adapted step-by-step to this new situation. Besides, let us mention a specificity of the two-dimensional case: if Dirichlet conditions are imposed on the inclusions, logarithmic profiles increasing at infinity are involved and other techniques have to be employed. This will be the subject of a further work.

3. Preliminary Results

3.1. *Scaling of Sobolev norms on parameter dependent domains*

On the trace space of parametrized domains. In the following, we will have to use the Sobolev space $H^{1/2}$ of the boundary of an ε-dependent domain Ω_ε. This space can be defined in two ways: either as $TH^1(\Omega_\varepsilon)$ the trace space of $H^1(\Omega_\varepsilon)$ with the norm

$$\|f\|_{TH^1(\Omega_\varepsilon)} = \inf\{\|u\|_{H^1(\Omega_\varepsilon)} |\ u \in H^1(\Omega_\varepsilon) \text{ with } u = f \text{ on } \partial\Omega_\varepsilon\},$$

either through its usual definition of Sobolev space, i.e. subspace of $L^2(\partial\Omega_\varepsilon)$ with finite norm (known as *intrinsic norm*)

$$\|f\|_{H^{1/2}(\partial\Omega_\varepsilon)} = \|f\|_{L^2(\partial\Omega_\varepsilon)} + [f]_{2,\partial\Omega_\varepsilon},$$

with

$$[f]^2_{2,\partial\Omega_\varepsilon} = \iint_{\partial\Omega_\varepsilon\times\partial\Omega_\varepsilon} \frac{|f(x) - f(y)|^2}{|x - y|^2}\, d\sigma_x d\sigma_y.$$

Gagliardo has shown in Ref. 8 that, if the domain is Lipschitz, the two different norms on $H^{1/2}$ are equivalent. For a family of domains parametrized by ε, this means that the domains should be uniformly Lipschitz with respect to ε.

Maz'ya and Poborchi discuss in Ref. 14, Sec. 4.1.3 situations where this property is violated. Consider a single interior perturbation $\Omega_\varepsilon = \Omega \backslash \varepsilon\overline{\omega}$ where the nucleation center 0 belongs to Ω. The two terms in the intrinsic norm should be weighted. We quote their result once adapted to the space $H^{1/2}$ in the case of plane domains: the trace norm $\|f\|_{TH^1(\Omega_\varepsilon)}$ is equivalent, uniformly in ε, to the norm

$$(\varepsilon|\ln\varepsilon|)^{-1/2}\|f\|_{L^2(\partial\Omega_\varepsilon)} + [f]_{2,\partial\Omega_\varepsilon}. \tag{3.1}$$

In this work, we use both definitions of the norm on $H^{1/2}(\partial\Omega_\varepsilon)$: the definition as $\|\cdot\|_{TH^1(\Omega_\varepsilon)}$ is involved in *a priori* estimates, and the intrinsic definition as $\|\cdot\|_{H^{1/2}(\partial\Omega_\varepsilon)}$ is used to compute the norm of explicit functions. Therefore, we only need a rough inequality allowing to control the $TH^1(\Omega_\varepsilon)$ norm with respect to ε for the family of deformations under consideration.

Lemma 3.1. *Let Ω_ε be defined in* (1.3). *There is a constant C* (*independent of* ε) *such that for all $f \in H^{1/2}(\partial\Omega_\varepsilon)$*

$$\|f\|_{TH^1(\Omega_\varepsilon)} \le C\varepsilon^{-2}\|f\|_{H^{1/2}(\partial\Omega_\varepsilon)}. \tag{3.2}$$

Proof. Fix $\varepsilon_0 > 0$ small enough and consider Ω_{ε_0}: this is a Lipschitz domain. By classical results, there is a continuous extension operator $E_{\varepsilon_0} : H^{1/2}(\partial\Omega_{\varepsilon_0}) \to H^1(\Omega_{\varepsilon_0})$.

Now, we define a diffeomorphism Φ_ε from Ω_{ε_0} onto Ω_ε involving three different scales:

- Φ_ε coincides with identity at scale 1 (in particular the boundary of Ω_0 is invariant),
- is a contraction of ratio $(\varepsilon/\varepsilon_0)^\alpha$ around 0,
- is a contraction of ratio $\varepsilon/\varepsilon_0$ around $x^\pm_{\varepsilon_0}$.

We construct, in two steps, such an application thanks to cutoff functions. Let us introduce some notation. Let $R, R_\pm > 0$ be such that

$$B(0, 2R) \subset \Omega_{\varepsilon_0}, \quad \omega^\pm_{\varepsilon_0} \subset B(x^\pm_{\varepsilon_0}, R_\pm) \subset B(0, R) \quad \text{and} \quad 0 \notin B(x^\pm_{\varepsilon_0}, 2R_\pm).$$

Now let φ be a non-increasing function in $\mathcal{C}^\infty([0, +\infty), [0, 1])$ with $\varphi(t) = 1$ if $t < 1$ and $\varphi(t) = 0$ if $t > 2$. The application $\Phi_{\varepsilon_0\to\varepsilon^\alpha}$ defined by

$$\Phi_{\varepsilon_0\to\varepsilon^\alpha}(x) = \left[1 - \varphi\left(\frac{|x|}{R}\right)\right]x + \varphi\left(\frac{|x|}{R}\right)\left(\frac{\varepsilon}{\varepsilon_0}\right)^\alpha x,$$

is a diffeomorphism that corresponds to identity outside the ball $B(0, 2R)$ and to a contraction around 0 of ratio $(\varepsilon/\varepsilon_0)^\alpha$ inside $B(0, R)$. Thus it maps $x^\pm_{\varepsilon_0}$ onto $x^\pm_\varepsilon$ and preserves 0. Note that $\Phi_{\varepsilon_0\to\varepsilon^\alpha}(B(x^\pm_{\varepsilon_0}, R_\pm)) = B(x^\pm_\varepsilon, R_\pm(\varepsilon/\varepsilon_0)^\alpha)$. In a similar way, we define $\Phi_{\varepsilon^\alpha\to\varepsilon}$ by

$$\begin{aligned}\Phi_{\varepsilon^\alpha\to\varepsilon}(x) &= \left[1-\varphi\left(\left(\frac{\varepsilon_0}{\varepsilon}\right)^\alpha\frac{|x-x_\varepsilon^-|}{R_-}\right)-\varphi\left(\left(\frac{\varepsilon_0}{\varepsilon}\right)^\alpha\frac{|x-x_\varepsilon^+|}{R_+}\right)\right]x\\ &\quad+\varphi\left(\left(\frac{\varepsilon_0}{\varepsilon}\right)^\alpha\frac{|x-x_\varepsilon^-|}{R_-}\right)\left[x_\varepsilon^-+\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-\alpha}(x-x_\varepsilon^-)\right]\\ &\quad+\varphi\left(\left(\frac{\varepsilon_0}{\varepsilon}\right)^\alpha\frac{|x-x_\varepsilon^+|}{R_+}\right)\left[x_\varepsilon^++\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-\alpha}(x-x_\varepsilon^+)\right].\end{aligned}$$

This is also a diffeomorphism that introduces the third scale. The wanted mapping is obtained by the composition $\Phi_\varepsilon = \Phi_{\varepsilon^\alpha\to\varepsilon}\circ\Phi_{\varepsilon_0\to\varepsilon^\alpha}$ and maps $\omega_{\varepsilon_0}^\pm$ onto $\omega_\varepsilon^\pm$. One checks that $\Phi_\varepsilon(\Omega_{\varepsilon_0})=\Omega_\varepsilon$ and that $\|\Phi_\varepsilon^{-1}\|_{W^{1,\infty}}\le C\varepsilon^{-1}$.

Now, thanks to E_{ε_0} and Φ_ε, we define an extension operator E_ε from $H^{1/2}(\partial\Omega_\varepsilon)$ into $H^1(\Omega_\varepsilon)$ by

$$E_\varepsilon(f) = \left[E_{\varepsilon_0}(f\circ\Phi_\varepsilon)\right]\circ\Phi_\varepsilon^{-1}.$$

From the definition of the trace norm, we check that

$$\|f\|_{H^{1/2}(\partial\Omega_\varepsilon)} \le \|E_\varepsilon(f)\|_{H^1(\Omega_\varepsilon)} \le C\|\Phi_\varepsilon^{-1}\|_{W^{1,\infty}}\left\|E_{\varepsilon_0}(f\circ\Phi_\varepsilon)\right\|_{H^1(\Omega_{\varepsilon_0})}.$$

Since E_{ε_0} is a continuous operator, there exists a constant c such that

$$\|E_{\varepsilon_0}(f\circ\Phi_\varepsilon)\|_{H^1(\Omega_{\varepsilon_0})} \le c\|f\circ\Phi_\varepsilon\|_{H^{1/2}(\partial\Omega_{\varepsilon_0})}.$$

Besides Φ_ε behaving like a contraction of ratio $\varepsilon/\varepsilon_0$ in the vicinity of $\partial\omega_{\varepsilon_0}^\pm$, we check that

$$\|f\circ\Phi_\varepsilon\|_{H^{1/2}(\partial\Omega_{\varepsilon_0})} \le \frac{\varepsilon_0}{\varepsilon}\|f\|_{H^{1/2}(\partial\Omega_\varepsilon)}, \tag{3.3}$$

since

$$\|f\circ\Phi_\varepsilon\|_{L^2(\partial\Omega_{\varepsilon_0})} \le \frac{\varepsilon_0}{\varepsilon}\|f\|_{L^2(\partial\Omega_\varepsilon)} \quad\text{and}\quad [f\circ\Phi_\varepsilon]_{2,\partial\Omega_{\varepsilon_0}} \le [f]_{2,\partial\Omega_\varepsilon}.$$

Gathering these estimates, we deduce (3.2). □

Remark 3.1. The weight arising in Lemma 3.1 is clearly not optimal, see (3.3). In particular, we have lost the dependency in α. Nevertheless, we do not need an equivalent norm of $\|\cdot\|_{TH^1(\Omega_\varepsilon)}$ for our purpose since a coarse estimate is enough to validate the complete asymptotic expansion.

Remark 3.2. The case (b) is a direct adaptation of the interior case: the boundary perturbation appears in a flat part of $\partial\Omega_0$ and this line is locally invariant under contraction.

Traces of smooth functions. In the following, we will face the question of evaluating on $\varepsilon\omega$ (with $\omega=\omega^\pm$) various norms of the trace of a function f which is smooth around 0.

Lemma 3.2. *Let f be a smooth function defined around* 0. *Let $M \geq 0$ be such that for all multi-indices k with $|k| < M$, $\partial_k f(0) = 0$. Let ω be a regular domain. Then,*

$$\|f\|_{H^{1/2}(\varepsilon\partial\omega)} \leq C\varepsilon^M, \tag{3.4}$$

$$\|f\|_{TH^1(\varepsilon\omega)} \leq C\varepsilon^{M-2}. \tag{3.5}$$

Proof. Let us first consider the L^2 norm. We set $x = \varepsilon X$, then

$$\|f\|^2_{L^2(\varepsilon\partial\omega)} = \int_{\varepsilon\partial\omega} |f(x)|^2 d\sigma_x = \varepsilon \int_{\partial\omega} |f(\varepsilon X)|^2 d\sigma_X.$$

Therefore, since f is assumed to be smooth around 0, its Taylor expansion provides the expansion $f(\varepsilon X) = \varepsilon^M P_M(X) + o(\varepsilon^M)$ (here P_M denotes the polynomial term of order M in the Taylor expansion of f at 0), then

$$\|f\|^2_{L^2(\varepsilon\partial\omega)} \leq C\varepsilon^{2(M+1)}. \tag{3.6}$$

We now consider the double integral term defining the fractional part of the norm $H^{1/2}$. By change of variables, we get

$$\begin{aligned}\iint_{(\varepsilon\partial\omega)\times(\varepsilon\partial\omega)} \frac{|f(x) - f(y)|^2}{|x-y|^2}\, d\sigma_x d\sigma_y &= \varepsilon^2 \iint_{\partial\omega\times\partial\omega} \frac{|f(\varepsilon X) - f(\varepsilon Y)|^2}{|\varepsilon X - \varepsilon Y|^2}\, d\sigma_X d\sigma_Y \\ &= \iint_{\partial\omega\times\partial\omega} \frac{|f(\varepsilon X) - f(\varepsilon Y)|^2}{|X - Y|^2}\, d\sigma_X d\sigma_Y.\end{aligned}$$

Now we have $|f(\varepsilon X) - f(\varepsilon Y)| \leq \varepsilon |X - Y| \|\nabla f\|_{L^\infty(\varepsilon\omega)}$. From Taylor's expansion of f at 0, we obtain easily $\|\nabla f\|_{L^\infty(\varepsilon\omega)} \leq C\varepsilon^{M-1}$. Then,

$$\iint_{\partial\omega\times\partial\omega} \frac{|f(\varepsilon X) - f(\varepsilon Y)|^2}{|X - Y|^2}\, d\sigma_X d\sigma_Y \leq C\varepsilon^{2M}.$$

By definition of the $H^{1/2}$ norm, we get the upper bound $\|f\|_{H^{1/2}(\varepsilon\partial\omega)} \leq C\varepsilon^M$. □

Lemma 3.3. *Let ω be a regular domain. Then, for $f \in H^1(\omega)$ with $\Delta f \in L^2(\omega)$,*

$$\|\partial_{\mathbf{n}} f\|_{H^{-1/2}(\varepsilon\partial\omega)} \leq \frac{1}{\varepsilon} \|\partial_{\mathbf{n}} F\|_{H^{-1/2}(\partial\omega)},$$

where F is deduced from f by dilation and $H^{-1/2}$ is equipped with the dual norm.

Proof. Let $\varphi \in H^{1/2}(\varepsilon\partial\omega)$. Using the scaling $X = \frac{x}{\varepsilon}$ and denoting $F(X) = f(x)$, $\Phi(X) = \varphi(x)$, we get by the Green formula

$$\begin{aligned}\int_{\varepsilon\partial\omega} \partial_{\mathbf{n}} f(x)\varphi(x) d\sigma_x &= \int_{\varepsilon\omega} \nabla f(x) \cdot \nabla\varphi(x) dx - \int_{\varepsilon\omega} \Delta f(x)\varphi(x) dx \\ &= \int_{\omega} \nabla F(X) \cdot \nabla\Phi(X) dX - \int_{\omega} \Delta F(X)\Phi(X) dX \\ &= \int_{\partial\omega} \partial_{\mathbf{n}} F(X)\Phi(X) d\sigma_X.\end{aligned}$$

We deduce

$$\sup_{\varphi\in H^{1/2}(\varepsilon\partial\omega)} \frac{\int_{\varepsilon\partial\omega}\partial_{\mathbf{n}} f(x)\varphi(x)d\sigma_x}{\|\varphi\|_{H^{1/2}(\varepsilon\partial\omega)}} = \sup_{\varphi\in H^{1/2}(\varepsilon\partial\omega)} \frac{\int_{\partial\omega}\partial_{\mathbf{n}} F(X)\Phi(X)d\sigma_X}{\|\Phi\|_{H^{1/2}(\partial\omega)}}\frac{\|\Phi\|_{H^{1/2}(\partial\omega)}}{\|\varphi\|_{H^{1/2}(\varepsilon\partial\omega)}}$$
$$\leq \frac{C}{\varepsilon}\sup_{\Phi\in H^{1/2}(\partial\omega)} \frac{\int_{\partial\omega}\partial_{\mathbf{n}} F(X)\Phi(X)d\sigma_X}{\|\Phi\|_{H^{1/2}(\partial\omega)}},$$

according to (3.3). □

3.2. *Existence and behavior of the profiles*

We now consider the boundary value problem (1.2). Accurate informations about the behavior at infinity of the profiles are needed for the analysis of the asymptotic expansion. Accordingly we introduce a definition which expresses a behavior at infinity like $|X|^{-p}$.

Definition 3.1. Let $\mathcal{O}_\infty(|X|^{-p})$ be the set of functions $f\in L^2(\mathbb{R}^2\setminus\overline{\omega^\pm})$ such that, for any multi-indice $i\in\mathbb{N}^2$, there exists a positive constant C such that

$$|X|^{p+|i|}|\partial^i f(X)|\leq C,\quad \forall\, X\in\mathbb{R}^2\setminus\overline{\omega^\pm}.$$

A function V is homogeneous of order $-k$ if $V(\lambda X)=\lambda^{-k}V(X)$ for $X\in\mathbb{R}^2$ and $\lambda>0$. The following proposition gathers an existence and uniqueness result from Ref. 2 with an expansion at infinity obtained through Fourier series.

Proposition 3.1. (Interior case) *Let ω be a smooth bounded domain of $\mathbb{R}^2$ with $0\in\omega$. We assume that $g\in H^{-1/2}(\partial\omega)$ satisfies $\langle g,1\rangle_{H^{-1/2}\times H^{1/2}}=0$. Then the boundary value problem*

$$\begin{cases} -\Delta V=0 & \text{in } \mathbb{R}^2\setminus\overline{\omega},\\ \partial_{\mathbf{n}} V=g & \text{on } \partial\omega,\\ V\to 0 & \text{at infinity,}\end{cases}\tag{3.7}$$

admits a unique weak solution V_0 in the variational space

$$\left\{V;\ \nabla V\in L^2(\mathbb{R}^2\setminus\overline{\omega}) \text{ and } \frac{V}{(1+|X|)\log(2+|X|)}\in L^2(\mathbb{R}^2\setminus\overline{\omega})\right\}.$$

Furthermore, its solution can be decomposed as

$$V_0(X)=\sum_{k=1}^{n}V_{0,k}(X)+\mathcal{O}_\infty(|X|^{-n+1}),\tag{3.8}$$

where $V_{0,k}\in\mathcal{O}_\infty(|X|^{-k})$ is a homogeneous harmonic function of order $-k$.

The corresponding result for a perturbation on the boundary is quoted from Refs. 6 and 7.

Proposition 3.2. (Boundary perturbations) *Let $\breve{\omega}^+$ be the perturbed upper half plane appearing in* (2.2). *Let f in $H^{1/2}(\partial\breve{\omega}^+)$ be such that $f=0$ on the two infinite connected half lines of $\partial\breve{\omega}^+$ (that is to say outside of the perturbation). Then the boundary value problem*

$$\begin{cases} -\Delta V = 0 & \textit{in } \breve{\omega}^+, \\ V = f & \textit{on } \partial\breve{\omega}^+, \\ V \to 0 & \textit{at infinity}, \end{cases} \tag{3.9}$$

admits a unique weak solution V_0^d in the variational space

$$\left\{ V;\ \nabla V \in L^2(\breve{\omega}^+) \textit{ and } \frac{V}{1+|X|} \in L^2(\breve{\omega}^+) \right\}.$$

Furthermore, this solution can be decomposed as

$$V_0^d(X) = \sum_{k=1}^{n} V_{0,k}(X) + \mathcal{O}_\infty(|X|^{-n+1}), \tag{3.10}$$

where $V_{0,k} \in \mathcal{O}_\infty(|X|^{-k})$ is a homogeneous harmonic function of order $-k$.

Remark 3.3. A homogeneous harmonic function of order $-k$ reads $r^{-k} f_k(\theta)$ where the radial function f_k is a linear combination of $\cos k\theta$ and $\sin k\theta$.

3.3. *Construction of the correctors*

In the sequel, we will use profiles to take into account the effect of $\omega_\varepsilon^\pm$ on $\omega_\varepsilon^\mp$. They have a small but nonvanishing trace on $\partial\Omega_\varepsilon$. In order to define the next corrector, we estimate their traces on the boundary $\partial\Omega_\varepsilon$.

Geometrical setting (a). We consider the traces on the other parts of $\partial\Omega_\varepsilon$, that is to say on $\partial\Omega_0$ and $\partial\omega_\varepsilon^\pm$. The expansion of $|x - x_\varepsilon^\pm|$ for $x \in \partial\Omega_0$ gives the existence of coefficients $a_l^\pm$ such that

$$|x - x_\varepsilon^\pm| = |x| \left(1 \mp \varepsilon^\alpha \frac{\mathbf{d}.x}{|x|^2} + \frac{\varepsilon^{2\alpha}}{|x|^2} \right)^{\frac{1}{2}} = \sum_{l \geq 0} a_l^\pm \varepsilon^{\alpha l}.$$

For any $x \in \partial\Omega_0$, we denote by $\theta_\varepsilon^\pm$ the angle of the polar coordinates centered at $x_\varepsilon^\pm$:

$$\cos\theta_\varepsilon^\pm = \frac{x_1 \mp d_1 \varepsilon^\alpha}{|x - x_\varepsilon^\pm|} \quad \text{and} \quad \sin\theta_\varepsilon^\pm = \frac{x_2 \mp d_2 \varepsilon^\alpha}{|x - x_\varepsilon^\pm|},$$

with (d_1, d_2) the coordinates of $\mathbf{d}$. Therefore, there exist coefficients $b_k^\pm$ such that

$$\theta_\varepsilon^\pm|_{\partial\Omega_0} = \sum_{k \geq 0} b_k^\pm \varepsilon^{\alpha k}.$$

Note that the leading terms $a_0^\pm$ and $b_0^\pm$ are nothing but the polar coordinates corresponding to the origin. For any normalized homogeneous harmonic function of order $-k$ of the decomposition (3.8), we deduce the expansion

$$\text{For } x \in \partial\Omega_0, \quad V_k\left(\frac{x - x_\varepsilon^\pm}{\varepsilon}\right) = \frac{\varepsilon^k}{|x - x_\varepsilon^\pm|^k} f_k(\theta_\varepsilon^\pm) = \varepsilon^k \sum_{l\geq 0} d_{0,l}^\pm \varepsilon^{\alpha l}. \tag{3.11}$$

Next, we examine the trace on $\partial\omega_\varepsilon^\mp$. Let x belong to $\partial\omega_\varepsilon^\mp$. There exists $X \in \partial\omega^\mp$ such that $x = x_\varepsilon^\mp + \varepsilon X$. Then, the distance between points x and $x_\varepsilon^\pm$ satisfies

$$\begin{aligned} |x - x_\varepsilon^\pm| &= |\mp 2\varepsilon^\alpha \mathbf{d} + \varepsilon X| = 2\varepsilon^\alpha \left(1 \mp \varepsilon^{1-\alpha} \mathbf{d} \cdot X + \frac{\varepsilon^{2(1-\alpha)}}{4} |X|^2\right)^{\frac{1}{2}} \\ &= \varepsilon^\alpha \sum_{l\geq 0} \tilde{a}_l^\pm \varepsilon^{(1-\alpha)l}. \end{aligned}$$

Here, the $\theta_\varepsilon^\pm$ admit the expansion

$$\theta_\varepsilon^\pm|_{\partial\omega_\varepsilon^\mp} = \sum_{k\geq 0} \tilde{b}_k^\pm \varepsilon^{(1-\alpha)k}.$$

The leading terms $\tilde{a}_0^\pm$ and $\tilde{b}_0^\pm$ satisfy $\tilde{a}_0^\pm = 2$, $\mathbf{d} = \mp(\cos\tilde{b}_0^\pm, \sin\tilde{b}_0^\pm)$. Therefore there exist coefficients $d_l^\pm$ entering into the expansion of the profile:

$$\text{For } x \in \partial\omega_\varepsilon^\mp, \quad V_k\left(\frac{x - x_\varepsilon^\pm}{\varepsilon}\right) = \sum_{l\geq k} d_l^\pm \varepsilon^{l(1-\alpha)}. \tag{3.12}$$

Geometrical setting (b). We perform the same analysis after splitting the outer boundary into the perturbed part and the unperturbed one. Namely we distinguish for $z \in \partial\Omega_\varepsilon \backslash \partial\omega_\varepsilon^-$ a neighboring part to 0 at distance of order ε and a far part containing the remaining boundary

$$V_k(z) = V_{k,n}(z) + V_{k,f}(z),$$

$$\text{with } V_{k,n}(z) = \left(1 - \zeta\left(\frac{z}{\varepsilon}\right)\right) V_k(z) \quad \text{and} \quad V_{k,f}(z) = \zeta\left(\frac{z}{\varepsilon}\right) V_k(z).$$

The same arguments as previously give an expansion of $V_{k,n}(z)$ in powers of $\varepsilon^{1-\alpha}$ starting with $\varepsilon^{k(1-\alpha)}$ as in (3.12).

4. Proofs of Theorems 2.1–2.2

4.1. *Proof of Theorem* 2.1

For the clarity of the presentation, we make a constructive proof to explain the ansatz. Let us now start with the asymptotic expansion and its first corrector. We introduce the first remainder r_ε^0 defined on Ω_ε by

$$u_\varepsilon = u_0 + r_\varepsilon^0.$$

Then r_ε^0 satisfies

$$\begin{cases} -\Delta r_\varepsilon^0 = 0 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^0 = 0 & \text{on } \partial\Omega_0, \\ \partial_{\mathbf{n}} r_\varepsilon^0 = -\partial_{\mathbf{n}} u_0 & \text{on } \partial\omega_\varepsilon^+ \cup \partial\omega_\varepsilon^-. \end{cases} \tag{4.1}$$

As mentioned in (1.2), we introduce the *profiles* $V_0^\pm$. Thanks to Proposition 3.1, they are the unique solutions of

$$\begin{cases} -\Delta V_0^\pm = 0 & \text{in } \mathbb{R}^2 \backslash \overline{\omega^\pm}, \\ \partial_{\mathbf{n}} V_0^\pm = -\mathbf{n} \cdot \nabla u_0(0) & \text{on } \partial\omega^\pm, \\ V_0^\pm \to 0 & \text{at infinity.} \end{cases} \tag{4.2}$$

Applying again Proposition 3.1, there exist $V_{0,k}^\pm \in \mathcal{O}_\infty(|X|^{-k})$ for $k = 1, \dots, N-1$ such that

$$V_0^\pm(X) = \sum_{k=1}^{N-1} V_{0,k}^\pm(X) + \mathcal{O}_\infty(|X|^{-N}), \quad \forall\, X \in \mathbb{R}^2 \backslash \overline{\omega^\pm}. \tag{4.3}$$

We now introduce the second remainder r_ε^1 defined for any $x \in \Omega_\varepsilon$ by:

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x).$$

Inserting this definition into the boundary value problem (1.4), we check that r_ε^1 satisfies

$$\begin{cases} -\Delta r_\varepsilon^1 = 0 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^1(x) = -\varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] & \text{for } x \in \partial\Omega_0, \\ \partial_{\mathbf{n}} r_\varepsilon^1(x) = \mathbf{n} \cdot \nabla u_0(0) - \mathbf{n} \cdot \nabla u_0(x) - \mathbf{n} \cdot \nabla V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) & \text{for } x \in \partial\omega_\varepsilon^+, \\ \partial_{\mathbf{n}} r_\varepsilon^1(x) = \mathbf{n} \cdot \nabla u_0(0) - \mathbf{n} \cdot \nabla u_0(x) - \mathbf{n} \cdot \nabla V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right) & \text{for } x \in \partial\omega_\varepsilon^-. \end{cases} \tag{4.4}$$

Let us give more information about the behavior of the trace of r_ε^1 on the boundaries. According to (4.3) and (4.4) the following relation holds for any $x \in \partial\Omega_0$:

$$r_\varepsilon^1(x) = -\varepsilon \sum_{k=1}^{N-1} \left[V_{0,k}^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_{0,k}^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + \varepsilon\mathcal{O}_\infty\left(\left|\frac{x}{\varepsilon}\right|^{-N}\right).$$

Then, using (3.11), there exist $f_{j,k}$ such that we can rewrite

$$r_\varepsilon^1(x) = \sum_{\substack{j \geq 1, k \geq 0, \\ j + \alpha k \leq N}} \varepsilon^{j+\alpha k} f_{j,k}(x) + o(\varepsilon^N), \quad \forall\, x \in \partial\Omega_0. \tag{4.5}$$

Let us look at the trace of r_ε^1 on $\partial\omega_\varepsilon^\pm$. For any $x \in \partial\omega_\varepsilon^\pm$, there exists $X \in \partial\omega^\pm$ such that:

$$x = x_\varepsilon^\pm + \varepsilon X = \pm\varepsilon^\alpha \mathbf{d} + \varepsilon X.$$

Thus

$$\nabla r_\varepsilon^1(x) = \nabla u_0(0) - \nabla u_0(\pm\varepsilon^\alpha \mathbf{d} + \varepsilon X) - \nabla V_0^\mp(\pm 2\varepsilon^{\alpha-1}\mathbf{d} + X). \tag{4.6}$$

Two contributions give the order of ∇r_ε^1 on $\partial\omega_\varepsilon^\pm$: the Taylor expansion of u_0 and the Neumann trace of the profiles $V_0^\mp$ on the inclusion $\partial\omega_\varepsilon^\pm$.

- Assuming u_0 is smooth enough, the Taylor expansion of ∇u_0 provides

$$\begin{aligned}&\nabla u_0(\pm\varepsilon^\alpha \mathbf{d} + \varepsilon X) - \nabla u_0(0)\\ &\quad = \sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{j+\alpha k}\frac{(\pm 1)^k}{(j+k)!} D^{j+k+1}u_0(0)[\mathbf{d}^k, X^j] + o(\varepsilon^N).\end{aligned}$$

  For convenience, we denote

$$g_{j,k}^\pm(X) = -\frac{(\pm 1)^k}{(j+k)!} D^{j+k+1}u_0(0)[\mathbf{d}^k, X^j]\cdot \mathbf{n}, \quad \forall X \in \partial\omega^\pm.$$

  Note that $D^{j+k+1}u_0(0)[\mathbf{d}^k, X^j]\cdot \mathbf{n}$ is harmonic as Taylor monomial function of the harmonic function u_0. Therefore one has

$$\int_{\partial\omega^\pm} g_{j,k}^\pm(X)d\sigma_X = 0. \tag{4.7}$$

- Since $\alpha < 1$, then $\varepsilon^{\alpha-1} \to \infty$ as $\varepsilon \to 0$ and so the coefficient $\varepsilon^{\alpha-1}\mathbf{d}$ gives the leading term in $\nabla V_0^\mp$. From Proposition 3.1, there exist $h_j^\mp$ satisfying:

$$\partial_{\mathbf{n}} V_0^\mp(\pm 2\varepsilon^{\alpha-1}\mathbf{d} + X) = \sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h_j^\mp(X) + o(\varepsilon^N), \tag{4.8}$$

  with

$$\int_{\partial\omega^\pm} h_j^\mp(X)d\sigma_X = 0. \tag{4.9}$$

Combining (4.6) and (4.8), we deduce for $x = \pm\varepsilon^\alpha\mathbf{d} + \varepsilon X \in \partial\omega_\varepsilon^\pm$

$$\partial_{\mathbf{n}} r_\varepsilon^1(x) = \sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{j+\alpha k} g_{j,k}^\pm(X) + \sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h_j^\mp(X) + o(\varepsilon^N). \tag{4.10}$$

Now we need to lift each boundary condition appearing in (4.5) and (4.10).

- The functions $f_{j,k}$ introduced in (4.5) generate correctors $F_{j,k}$ defined by

$$\begin{cases} -\Delta F_{j,k} = 0 & \text{in } \Omega_0, \\ F_{j,k} = -f_{j,k} & \text{on } \partial\Omega_0. \end{cases} \tag{4.11}$$

  These correctors do not satisfy the Neumann condition on the boundary of the inclusions $\partial\omega_\varepsilon^\pm$ and so generate errors on these boundaries.
- The functions $g_{j,k}^\pm$ and $h_j^\mp$ generate profiles $G_{j,k}^\pm$ and $H_j^\mp$ with same behavior as the first corrector. These profiles satisfy:

$$\begin{cases} -\Delta G_{j,k}^\pm = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega^\pm}, \\ \partial_{\mathbf{n}} G_{j,k}^\pm = -g_{j,k}^\pm & \text{on } \partial\omega^\pm, \\ G_{j,k}^\pm \to 0 & \text{at infinity}, \end{cases} \tag{4.12}$$

  and

$$\begin{cases} -\Delta H_j^\mp = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega^\pm}, \\ \partial_{\mathbf{n}} H_j^\mp = -h_j^\mp & \text{on } \partial\omega^\pm, \\ H_j^\mp \to 0 & \text{at infinity}. \end{cases} \tag{4.13}$$

  The compatibility conditions (4.7) and (4.9) ensure the existence of these profiles.

The third remainder is naturally defined by:

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + \varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + \sum_{\substack{j\geq 1, k\geq 0,\\ j+\alpha k\leq N}} \varepsilon^{j+\alpha k} F_{j,k}(x) \\ &\quad + \sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{1+j+\alpha k}\left[G_{j,k}^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + G_{j,k}^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] \\ &\quad + \sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{1+j-\alpha j}\left[H_j^+\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + H_j^-\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^2(x). \end{aligned} \tag{4.14}$$

We have defined new functions such that $\Delta r_\varepsilon^2 = 0$ in Ω_ε. There are three contributions to determine the following remainder of the asymptotic expansion by this way:

- The Dirichlet trace on $\partial\Omega_0$ comes from the trace of $G_{j,k}^\pm$ and $H_j^\pm$. To construct the following term for the asymptotic expansion, we have to lift this condition.
- The functions $F_{j,k}$ do not satisfy the Neumann condition on the boundary of the inclusions $\partial\omega_\varepsilon^\pm$ and we have to lift them as well.
- Finally we have a corrector due to the interaction: $G_{j,k}^+$ and H_j^+ satisfy the Neumann condition on $\partial\omega_\varepsilon^+$ but not on $\partial\omega_\varepsilon^-$ and similarly for $G_{j,k}^-$, H_j^-. This is the third condition to lift.

Remark 4.1. As a consequence of (4.14), we can obtain the asymptotic expansion of Theorem 2.2 for $N = 2$. Indeed, we just have to gather terms with same power of ε in both variables x and $\frac{x-x_\varepsilon^\pm}{\varepsilon}$. Up to order ε^2, we get

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + \varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] \\ &\quad + \sum_{(p,q)\in\mathcal{K}_2} \varepsilon^{p+\alpha q}\left(v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right]\right) \\ &\quad + o(\varepsilon^2), \end{aligned}$$

with

$$\begin{aligned} \mathcal{K}_2 &= \left\{(p,q)\in\mathbb{Z}^2 \;\middle|\; p\geq 0,\, q\geq -\frac{3}{2}p+1,\, q\geq -p \text{ and } p+\alpha q\leq 2\right\}, \\ v_{p+\alpha q} &= F_{p,q}, \\ V_{p+\alpha q}^\pm &= \begin{cases} G_{p-1,q}^\pm + H_{-q}^\pm & \text{if } p+q=1, \\ G_{p-1,q}^\pm & \text{otherwise.} \end{cases} \end{aligned}$$

The same recombination may be obtained at any order ε^N. Nevertheless, all the terms of steps up to $N-1$ (written with a remainder of size $o(\varepsilon^N)$) mix to give the contribution at step N. The general formula for $v_{p+\alpha q}$ and $V_{p+\alpha q}^\pm$ is too technical to be reproduced here. In the following, we mainly focus on the powers of ε involved in the expansion rather than describing the algorithmic construction of the terms.

The remainder r_ε^2 satisfies

$$\begin{cases} -\Delta r_\varepsilon^2 = 0 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^2(x) = -\displaystyle\sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{1+j+\alpha k}\left[G_{j,k}^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + G_{j,k}^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] & \\ \qquad - \displaystyle\sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{1+j-\alpha j}\left[H_j^+\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + H_j^-\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + o(\varepsilon^N) & \text{on } \partial\Omega_0, \\ \partial_{\mathbf{n}} r_\varepsilon^2(x) = -\displaystyle\sum_{\substack{j\geq 0, k\geq 1,\\ j+\alpha k\leq N}} \varepsilon^{j+\alpha k}\partial_{\mathbf{n}} F_{j,k}(x) + \displaystyle\sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{j+\alpha k}\partial_{\mathbf{n}} G_{j,k}^\mp\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) & \\ \qquad + \displaystyle\sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{j-\alpha j} H_j^\mp\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + o(\varepsilon^N) & \text{on } \partial\omega_\varepsilon^\pm. \end{cases} \tag{4.15}$$

Let us explain the evolution of the powers of ε in the construction of the asymptotic expansion. We write the possible powers of ε in the form $j + \alpha k$ with $(j,k) \in \mathbb{Z}^2$. First we look at r_ε^1. Powers of ε appearing in the Dirichlet trace on $\partial\Omega_0$ are

$$K_1^1 = \{(j,k) \in \mathbb{N}^2 \mid j \geq 1\}.$$

For the Neumann condition on the inclusions, we define two sets:

$$K_2^1 = \{(j,k) \in \mathbb{N}^2 \mid j + k \geq 1\},$$
$$K_3^1 = \{(j,k) \in \mathbb{N} \times \mathbb{Z} \mid j \geq 2 \text{ and } k = -j\}.$$

Finally, let K^1 be defined by:

$$K^1 = K_1^1 \cup K_2^1 \cup K_3^1 = K_2^1 \cup K_3^1,$$

since $K_1^1 \subset K_2^1$. The set K^1 can be rewritten as the intersection of three convex sets:

$$K^1 = \{j \geq 0\} \cap \left\{k \geq -\frac{3}{2}j + 1\right\} \cap \{k \geq -j\}.$$

Similarly, K^2 is the set of powers of ε appearing with the remainder r_ε^3. These powers come from combination of K_2^1 and K_3^1. Let us develop all the possible configurations:

- K_2^1 with K_2^1: The terms have the form $\varepsilon^{j+j'+\alpha(k+k')}$ with $(j,k,j',k') \in \mathbb{N}^4$, $j + k \geq 1$, $j' + k' \geq 1$. Then
$$K_{2,2}^2 = \{(j,k) \in \mathbb{N}^2 \mid j + k \geq 2\}.$$
- K_2^1 with K_3^1: This combination leads to terms of the form $\varepsilon^{j+j'+\alpha(k-j')}$ with $j + k \geq 1$, $j' \geq 2$, then
$$K_{2,3}^2 = \{(j,k) \in \mathbb{N} \times \mathbb{Z} \mid j \geq 2, j + k \geq 1\}.$$
- K_3^1 with K_3^1: This combination leads to the definition
$$K_{3,3}^2 = \{(j,k) \in \mathbb{N} \times \mathbb{Z} \mid j \geq 4, k = -j\}.$$

The set $K^2 = \cup_{2 \leq j \leq k \leq 3} K_{j,k}^2$ is drawn in Fig. 4. It can be written as the intersection of three convex sets:

$$K^2 = \{j \geq 0\} \cap \left\{k \geq -\frac{3}{2}j + 2\right\} \cap \{k \geq -j\}.$$

At each step of the construction of the asymptotic expansion, we obtain a new set K^n of the possible powers of ε. Let us look at the evolution of the set K^n with $n \geq 2$. We can sum up the possibilities in three combinations:

- Two terms of the form $\varepsilon^{j+\alpha k}$;
- Two terms of the form $\varepsilon^{j-\alpha j}$;
- One term of the form $\varepsilon^{j+\alpha k}$ and one of the form $\varepsilon^{j-\alpha j}$.

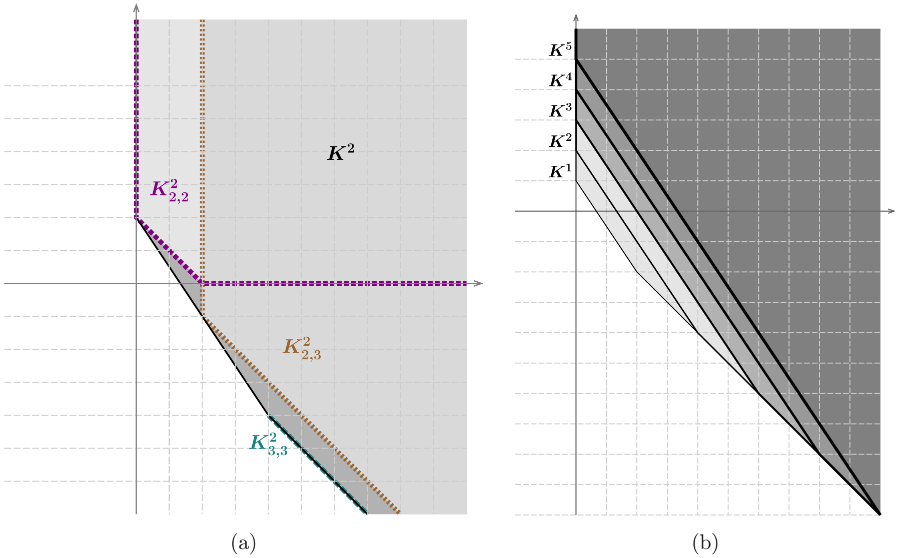


Fig. 4. The sets of indices. (a) Set K^2, (b) Set K^j.

To deduce the set K^3, we start from K^2 and we can make two operations: a translation $j\mathbf{e}_1 \oplus k\mathbf{e}_2$ with $j+k \geq 1$ or a translation $j(\mathbf{e}_1 - \mathbf{e}_2)$ with $j \geq 2$. Then K^3 is the convex defined by

$$K^3 = \{j \geq 0\} \cap \left\{k \geq -\frac{3}{2}j + 3\right\} \cap \{k \geq -j\}.$$

By induction, we obtain immediately that the leading terms for the Laplacian and traces of the remainder of order n are of the form $\mathcal{O}(\varepsilon^{j+\alpha k})$ with $(j,k) \in K^n$ defined by

$$K^n = \{j \geq 0\} \cap \left\{k \geq -\frac{3}{2}j + n\right\} \cap \{k \geq -j\}.$$

The first sets K^n are represented in Fig. 4. The vertices of the convex set K^n are $(0,n)$ and $(2n,-2n)$. The leading term is then $\mathcal{O}(\varepsilon^{\min(\alpha n, 2n(1-\alpha))})$. Let us define the critical exponent α such that $\alpha n = 2n(1-\alpha)$, that is

$$\alpha_c = \frac{2}{3}.$$

Then, the leading term is $\mathcal{O}(\varepsilon^{\alpha n})$ if $\alpha < \alpha_c$ and $\mathcal{O}(\varepsilon^{2n(1-\alpha)})$ else. This expresses the fact that if the perturbations are rather close to each other ($\alpha_c < \alpha$), the leading term comes from the interaction (corresponding to the $n(1-\alpha)$ exponent), while it is given by the classical correctors induced by the Taylor expansion around 0 when the perturbations are distant enough ($\alpha < \alpha_c$). This was rather expectable from the physical intuition of the problem.

In order to justify the size of the remainder in this formal expansion, we apply the usual *a priori* estimates thanks to Lemmas 3.2 and 3.3. The obtained bound for $\|r_\varepsilon^n\|_{H^1(\Omega_\varepsilon)}$ is of order $\mathcal{O}(\varepsilon^{\min(\alpha n, 2n(1-\alpha))-2})$. We recover the optimal estimate writing $r_\varepsilon^n = r_\varepsilon^{n+\ell} + \mathcal{O}(\varepsilon^{\min(\alpha n, 2n(1-\alpha))})$ where $\ell = \max([\frac{2}{\alpha}], [\frac{1}{1-\alpha}])$.

We can determine the maximum number of times $n_{\max}$ we have to perform this iterative procedure to have an asymptotic expansion of order N:

$$n_{\max} = \begin{cases} \left\lceil \dfrac{N}{\alpha} \right\rceil & \text{if } \alpha \leq \alpha_c; \\ \left\lceil \dfrac{N}{2(1-\alpha)} \right\rceil & \text{if } \alpha > \alpha_c. \end{cases}$$

4.2. *Proof of Theorem* 2.2

We will explain the first two steps of the construction of the asymptotic expansion. The complete construction given in Sec. 4.1 for interior inclusions can easily be adapted here. The main difference comes from the new lift induced by the cutoff in the slow variables.

We split the exterior boundary of Ω_ε in two parts: $\Gamma_\varepsilon^+ = \varepsilon\Gamma^+$ (see Fig. 3) and $\Gamma_\varepsilon^0 = \partial\Omega_\varepsilon \backslash (\partial\omega_\varepsilon^- \cup \Gamma_\varepsilon^+)$. We consider two smooth cutoff functions ζ and χ defined on $\mathbb{R}^+$ such that $\zeta(r)$ vanishes for $r < r_\bullet$ and $\zeta(r) = 1$ for $r > r^\bullet$, and $\chi(r) = 1$ for $r < r_*$ and $\chi(r) = 0$ for $r > r^*$. The Taylor expansion of u_0 at order N reads

$$u_0(x) = \chi(|x|) \sum_{k=0}^{N} a_k x^k + R_N(x) = \chi(|x|) T_N(x) + R_N(x).$$

The function u_0 is not necessarily defined in the whole domain Ω_ε but its Taylor expansion T_N can be extended to Ω_ε. Hence we define the truncated function belonging to $H^1(\Omega_\varepsilon)$

$$\tilde{u}_0(x) = \chi(|x|) T_N(x) + \zeta\left(\left|\frac{x}{\varepsilon}\right|\right) R_N(x).$$

The difference $\tilde{u}_0 - u_0$ is of order ε^N (see Refs. 7 and 17). The first remainder r_ε^0 defined on Ω_ε by

$$u_\varepsilon = \tilde{u}_0 + r_\varepsilon^0,$$

satisfies

$$\begin{cases} -\Delta r_\varepsilon^0 = \varphi_\varepsilon^0 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^0 = 0 & \text{on } \Gamma_\varepsilon^0, \\ r_\varepsilon^0 = -\tilde{u}_0 & \text{on } \Gamma_\varepsilon^+, \\ \partial_{\mathbf{n}} r_\varepsilon^0 = -\mathbf{n} \cdot \nabla \tilde{u}_0 & \text{on } \partial\omega_\varepsilon^-. \end{cases} \tag{4.16}$$

To define φ_ε^0, we rewrite r_ε^0:

$$\begin{aligned} r_\varepsilon^0 &= (u_\varepsilon - u_0) - (\tilde{u}_0 - u_0) \\ &= (u_\varepsilon - u_0) - \left(\zeta\left(\left|\frac{\cdot}{\varepsilon}\right|\right) - 1\right)(u_0 - \chi(|\cdot|)T_N). \end{aligned}$$

Since T_N is harmonic and considering the intersection of the support of the cutoff functions χ and ζ, we get for ε small enough

$$\begin{aligned} \varphi_\varepsilon^0 &= \frac{1}{\varepsilon^2}\Delta\zeta\left(\left|\frac{\cdot}{\varepsilon}\right|\right)(u_0 - \chi(|\cdot|)T_N) - \frac{2}{\varepsilon}\nabla\zeta\left(\left|\frac{\cdot}{\varepsilon}\right|\right)\cdot\nabla(u_0 - \chi(|\cdot|)T_N) \\ &= \mathcal{O}(\varepsilon^{N-1}). \end{aligned}$$

This contribution is small enough to be incorporated in the remainder r_ε^N of the target expansion.

We easily check that u_0 and $\tilde{u}_0$ equal 0 on Γ_ε^0. We introduce the *profiles* $V_0^\pm$. According to Proposition 3.1, there exists a unique solution V_0^- of

$$\begin{cases} -\Delta V_0^- = 0 & \text{in } \mathbb{R}^2\setminus\overline{\omega^-}, \\ \partial_{\mathbf{n}} V_0^- = -\mathbf{n}\cdot\nabla u_0(0) & \text{on } \partial\omega^-, \\ V_0^- \to 0 & \text{at infinity.} \end{cases} \tag{4.17}$$

Proposition 3.2 gives the existence and uniqueness of the solution V_0^+ of the problem:

$$\begin{cases} -\Delta V_0^+ = 0 & \text{in } \breve{\omega}^+, \\ V_0^+(X) = -\nabla u_0(0)\cdot X & \text{on } \Gamma^+, \\ V_0^+ = 0 & \text{on } \partial\breve{\omega}^+\setminus\Gamma^+, \\ V_0^+ \to 0 & \text{at infinity.} \end{cases} \tag{4.18}$$

We are now ready to introduce the second remainder r_ε^1:

$$u_\varepsilon(x) = \tilde{u}_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\left(\frac{x}{\varepsilon}\right)\right] + r_\varepsilon^1(x).$$

Since $\chi \equiv 1$ on $\partial\omega_\varepsilon^- \cup \Gamma_\varepsilon^+$, we check that r_ε^1 satisfies

$$\begin{cases} -\Delta r_\varepsilon^1 = \varphi_\varepsilon^0 + \varphi_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^1(x) = -\varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\left(\frac{x}{\varepsilon}\right)\right] & \text{for } x \in \Gamma_\varepsilon^0, \\ r_\varepsilon^1(x) = -\tilde{u}_0(x) - \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x}{\varepsilon}\right)\right] & \text{for } x \in \Gamma_\varepsilon^+, \\ \partial_{\mathbf{n}} r_\varepsilon^1(x) = \mathbf{n}\cdot\nabla u_0(0) - \mathbf{n}\cdot\nabla\tilde{u}_0(x) - \mathbf{n}\cdot\nabla V_0^+\left(\frac{x}{\varepsilon}\right) & \text{for } x \in \partial\omega_\varepsilon^-. \end{cases} \tag{4.19}$$

Here

$$\varphi^1_\varepsilon(x) = \varepsilon\Delta(\chi(|x|)V_0^+(x)) = \nabla\chi(|x|)\cdot\nabla V_0^+\Big(\frac{x}{\varepsilon}\Big) + \varepsilon\Delta\chi(|x|)V_0^+\Big(\frac{x}{\varepsilon}\Big).$$

There exist f_j such that

$$\varphi^1_\varepsilon(x) = \sum_{2\le j\le N-1} \varepsilon^j f_j(x) + \mathcal{O}_\infty\Big(\Big|\frac{x}{\varepsilon}\Big|^{-N}\Big).$$

According to Propositions 3.1 and 3.2, we can find homogeneous functions $V^\pm_{0,k}$ such that for any $x \in \Gamma^0_\varepsilon$:

$$r^1_\varepsilon(x) = -\varepsilon \sum_{1\le k\le N-1}\Big[V^-_{0,k}\Big(\frac{x - x^-_\varepsilon}{\varepsilon}\Big) + \chi(|x|)V^+_{0,k}\Big(\frac{x}{\varepsilon}\Big)\Big] + \varepsilon\mathcal{O}_\infty\Big(\Big|\frac{x}{\varepsilon}\Big|^{-N}\Big).$$

Using (3.11), there exist $f_{j,k}$ such that we can rewrite on Γ^0_ε

$$r^1_\varepsilon(x) = \sum_{\substack{j\ge 1, k\ge 0,\\ j+\alpha k\le N}} \varepsilon^{j+\alpha k} f_{j,k}(x) + \mathcal{O}(\varepsilon^N). \tag{4.20}$$

Let $x \in \Gamma^+_\varepsilon$ and $X \in \Gamma^+$ be such that $x = \varepsilon X$. We define g^+_j on $\partial\breve{\omega}^+$ by

$$g^+_j(X) = -\frac{1}{j!}D^j u_0(0)[X^{(j)}] \quad \text{if } X \in \Gamma^+,$$

and by 0 elsewhere so that

$$\tilde{u}_0(\varepsilon X) + \varepsilon V_0^+(X) = \sum_{2\le j\le N} \varepsilon^k g^+_j(X).$$

There exist h^-_j coming from the trace of V_0^- such that

$$r^1_\varepsilon(x) = -\sum_{2\le j\le N}\varepsilon^k g^+_j(X) - \varepsilon\sum_{1\le j\le \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h^-_j(X) + o(\varepsilon^N). \tag{4.21}$$

Let us look at the Neumann condition on $\partial\omega^-_\varepsilon$. As $\chi \equiv 1$ on $\partial\omega^-_\varepsilon$, we have

$$\nabla r^1_\varepsilon(x) = \nabla u_\varepsilon(x) - \nabla\tilde{u}_0(x) - \nabla V_0^-\Big(\frac{x - x^-_\varepsilon}{\varepsilon}\Big) - \nabla V_0^+\Big(\frac{x}{\varepsilon}\Big).$$

Let $x \in \partial\omega^-_\varepsilon$, there exists $X \in \partial\omega^-$ such that $x = \varepsilon^\alpha \mathbf{d} + \varepsilon X$. Since

$$\nabla\tilde{u}_0(x) = \nabla T_N(x) + \nabla R_N(x) = \nabla u_0(x) + \mathcal{O}(\varepsilon^N),$$

a Taylor expansion of u_0 gives

$$\begin{aligned}\nabla u_0(0) - \nabla\tilde{u}_0(x) &= \nabla u_0(0) - \nabla u_0(\varepsilon^\alpha\mathbf{d} + \varepsilon X) + \mathcal{O}(\varepsilon^N)\\ &= -\sum_{\substack{j\ge 0, k\ge 0,\\ 0<j+\alpha k\le N}} \frac{\varepsilon^{j+\alpha k}}{(j+k)!} D^{j+k+1}u_0(0)[\mathbf{d}^{(k)}, X^{(j)}] + o(\varepsilon^N).\end{aligned}$$

We denote

$$g^-_{j,k}(X) = \frac{-1}{(j+k)!} D^{j+k+1}u_0(0)[\mathbf{d}^k, X^j]\cdot \mathbf{n}, \quad \forall X \in \partial\omega^-.$$

Let us analyze now ∇V_0^+. Since $\frac{x}{\varepsilon} = \varepsilon^{\alpha-1}\mathbf{d} + X$, Proposition 3.2 gives the existence of coefficients h_j^+ such that, on $\partial\omega_\varepsilon^-$,

$$\partial_{\mathbf{n}} V_0^+(\varepsilon^{\alpha-1}\mathbf{d} + X) = \sum_{2\le j\le \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h_j^+(X) + o(\varepsilon^N).$$

Consequently, on $\partial\omega_\varepsilon^-$,

$$\partial_{\mathbf{n}} r_\varepsilon^1(x) = \sum_{\substack{j\ge 0, k\ge 0,\\ 0<j+\alpha k\le N}} \varepsilon^{j+\alpha k} g^-_{j,k}(X) - \sum_{2\le j\le \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h_j^+(X) + o(\varepsilon^N). \tag{4.22}$$

To construct the following corrector r_ε^2, we define w_ε^j as the solution in $H^1(\Omega_\varepsilon)$ of

$$\begin{cases} -\Delta w_\varepsilon^j = -f_j & \text{in } \Omega_\varepsilon, \\ w_\varepsilon^j = 0 & \text{on } \partial\Omega_0. \end{cases}$$

We have to fit each boundary conditions given in (4.20) for Γ_ε^0, (4.21) for Γ_ε^+ and (4.22) for $\partial\omega_\varepsilon^-$. The functions $f_{j,k}$ introduced in (4.20) generate correctors $F_{j,k}$ defined by

$$\begin{cases} -\Delta F_{j,k} = 0 & \text{in } \Omega_0, \\ F_{j,k} = -f_{j,k} & \text{on } \partial\Omega_0. \end{cases} \tag{4.23}$$

The functions g_j^+, $g^-_{j,k}$ and $h_j^\mp$ generate profiles G_j^+, $G^-_{j,k}$ and $H_j^\mp$ with similar behavior as the other correctors. These profiles satisfy:

$$\begin{cases} -\Delta G_j^+ = 0 & \text{in } \breve{\omega}^+, \\ G_j^+ = -g_j^+ & \text{on } \partial\breve{\omega}^+, \\ G_j^+ \to 0 & \text{at infinity}, \end{cases} \qquad \begin{cases} -\Delta H_j^- = 0 & \text{in } \breve{\omega}^+, \\ H_j^- = -h_j^- & \text{on } \partial\breve{\omega}^+, \\ H_j^- \to 0 & \text{at infinity}, \end{cases}$$

and

$$\begin{cases} -\Delta G^-_{j,k} = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega^-}, \\ \partial_{\mathbf{n}} G^-_{j,k} = -g^-_{j,k} & \text{on } \partial\omega^-, \\ G^-_{j,k} \to 0 & \text{at infinity}, \end{cases} \qquad \begin{cases} -\Delta H_j^+ = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega^-}, \\ \partial_{\mathbf{n}} H_j^+ = -h_j^+ & \text{on } \partial\omega^-, \\ H_j^+ \to 0 & \text{at infinity}. \end{cases}$$

We check the compatibility conditions ensuring the existence of such profiles. The following steps are similar to those in the case of interior inclusions and we can make the same analysis for the indices appearing in the asymptotic expansion.

5. Numerical Experiments

The computation of the solution u_ε of problem (1.4) is not a straightforward problem since a very fine mesh is required if ε is small. For such values of ε, it is natural to use the asymptotic expansions presented in Theorems 2.1–2.2. Precisely, we approximate u_ε by its first order expansion

$$u_1(x) = u_0(x) + \varepsilon \left[V_0^- \left(\frac{x - x_\varepsilon^-}{\varepsilon} \right) + V_0^+ \left(\frac{x - x_\varepsilon^+}{\varepsilon} \right) \right]. \tag{5.1}$$

This means that u_0 and the profiles $V_0^\pm$ have to be computed. While u_0 is the solution of a classical boundary value problem (in an ε-independent domain which may be coarsely meshed), the profiles are solution of a problem posed on an infinite domain. We present in Sec. 5.1 a numerical method to obtain an accurate approximation of the profiles, and in Sec. 5.2, we show how it is used to compute an approximation of u_ε.

The numerical results shown hereafter have been performed with the Finite Element Library Mélina, Ref. 11.

5.1. *Computation of the profiles*

In order to compute the profiles $V_0^\pm$ involved in formula (5.1), we introduce the *normalized vectorial profile* $\mathbf{V} = \mathbf{V}_\omega$, solution of the following exterior boundary value problem

$$\begin{cases} -\Delta \mathbf{V} = 0 & \text{in } \mathbb{R}^2 \backslash \omega, \\ \partial_{\mathbf{n}} \mathbf{V} = \mathbf{g} & \text{on } \partial\omega, \\ \mathbf{V} \to 0 & \text{at infinity,} \end{cases}$$

with $\mathbf{g} = -\mathbf{n}$. We can recover $V^\pm$ from $\mathbf{V}_{\omega^\pm}$ via the formula

$$V^\pm = \nabla u_0(0) \cdot \mathbf{V}_{\omega^\pm},$$

so that formula (5.1) reads

$$u_1(x) = u_0(x) + \varepsilon \nabla u_0(0) \cdot \left[\mathbf{V}_{\omega^-} \left(\frac{x - x_\varepsilon^-}{\varepsilon} \right) + \mathbf{V}_{\omega^+} \left(\frac{x - x_\varepsilon^+}{\varepsilon} \right) \right]. \tag{5.2}$$

The profile $\mathbf{V}$ will be approximated componentwise: V and g denote the first component of $\mathbf{V}$ and $\mathbf{g}$, respectively (of course, the same can be done for the second component). Several approaches are available to compute V: integral equation, infinite elements, truncated domain with integral representation or artificial boundary condition. For the latter, we propose three absorbing conditions on $|x| = R$:

$$V = 0, \tag{5.3}$$

$$V + R\partial_{\mathbf{n}} V = 0, \tag{5.4}$$

$$V + \frac{3R}{2} \partial_{\mathbf{n}} V - \frac{R^2}{2} \Delta_\tau V = 0. \tag{5.5}$$

These conditions (Dirichlet/Robin/Ventcel) are said of order 0, 1 and 2, respectively (the Robin condition was already used in Ref. 7). The considered problem is then

$$\begin{cases} -\Delta V = 0 & \text{in } B(0,R)\setminus\omega, \\ \partial_{\mathbf{n}} V = g & \text{on } \partial\omega, \\ (5.3) \text{ or } (5.4) \text{ or } (5.5) & \text{on } \partial B(0,R). \end{cases}$$

We present here an alternative method based on a conformal mapping to convert the exterior domain into a bounded one. Precisely, we consider the inversion-symmetry $\varphi : z \mapsto 1/z$. The Laplace equation $-\Delta V = 0$ remains unchanged by homogeneity, the transformed profile $W = V \circ \varphi$ solves then the Neumann boundary value problem

$$\begin{cases} -\Delta W = 0 & \text{in } \varphi(\omega), \\ \partial_{\mathbf{n}} W = \partial_{\mathbf{s}}\varphi(g \circ \varphi) & \text{on } \partial\varphi(\omega), \\ W(0) = 0. \end{cases}$$

In the case where ω is the unit disk, the profile is explicitly known:

$$V(x) = \frac{\cos\theta}{r} = \frac{x_1}{x_1^2 + x_2^2} \quad \text{and} \quad W(x) = x_1.$$

Figure 6 presents the accuracy of the "inversion method" compared with the "artificial boundary method" (absorbing boundary condition of various order with cutoff radius $R = 10$; results shown for $g(x) = \cos\theta - 2\cos 2\theta - 3\cos 3\theta$ for which the exact solution is $V(x) = \cos\theta/r + \cos 2\theta/r^2 + \cos 3\theta/r^3$). The computations have been done on a fixed mesh for each method ($\mathbb{Q}_8$ geometric approximation, see Fig. 5), and the interpolation degree is increased from $\mathbb{Q}_1$ to $\mathbb{Q}_8$.

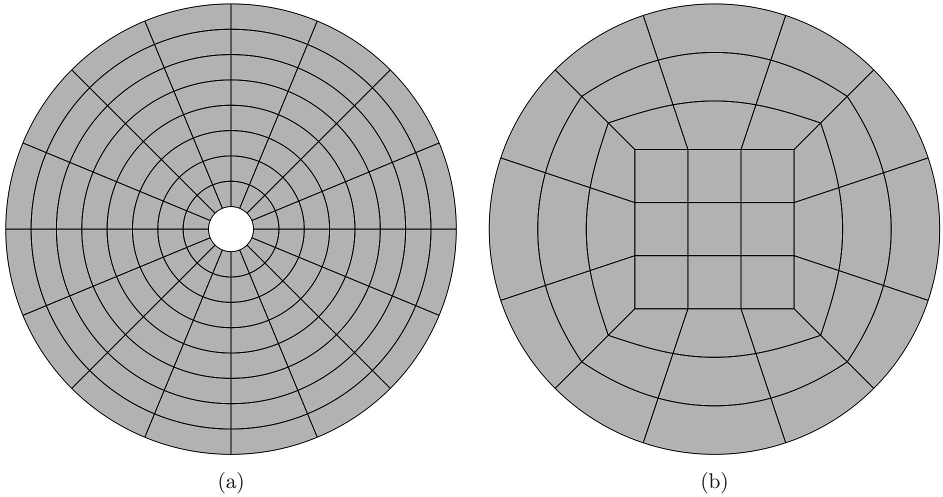


Fig. 5. Meshes used for the computation of the profiles. (a) Artificial boundary method ($R = 10$), (b) Inversion method.

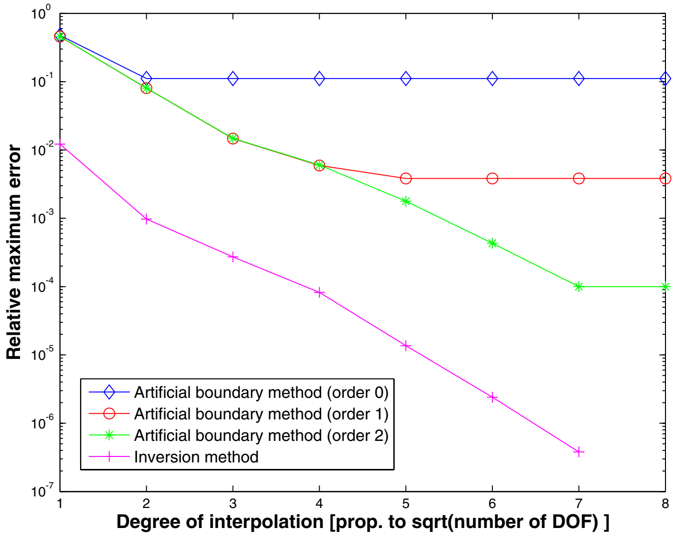
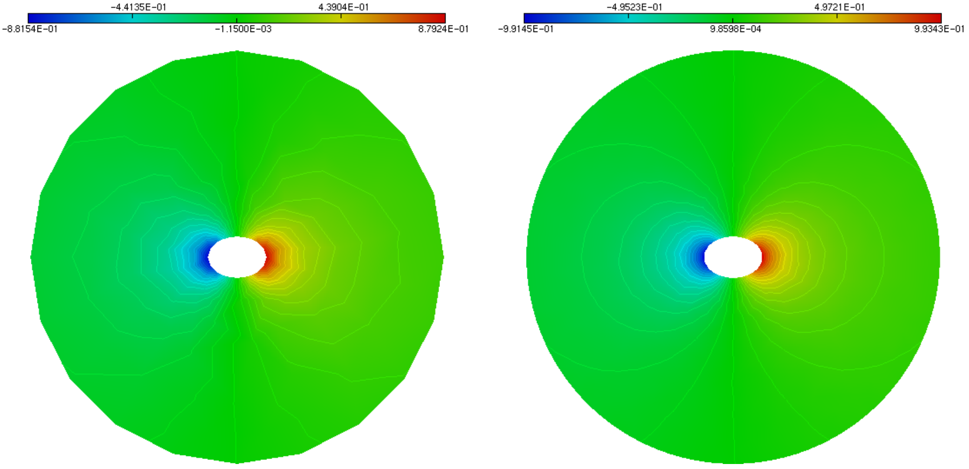


Fig. 6. Comparison inversion method/absorbing boundary condition.

It clearly appears that the artificial boundary method requires much more degrees of freedom than the inversion method. Let us mention that the cutoff radius $R = 10$ might be increased, this limiting factor is the cause of the locking observed in Fig. 6 for the absorbing boundary conditions.

Figure 7 shows the profile computed with both methods when ω is an ellipse: each computation involves $\mathbb{P}_1$-elements with 140 degrees of freedom (DOF) (the



(a) (b)

Fig. 7. Profile obtained for the same number of DOF. (a) Transparent boundary condition method, (b) Inversion approach.

solution obtained by inversion has been projected onto a fine mesh for comparison). It it clear that the inversion method provides a better accuracy for the same computation cost.

5.2. *Transfer and superposition*

The profiles computed above have to be mapped onto the grid where u_0 is defined to build the approximation u_1, see (5.1). This has been done via the following automatic procedure:

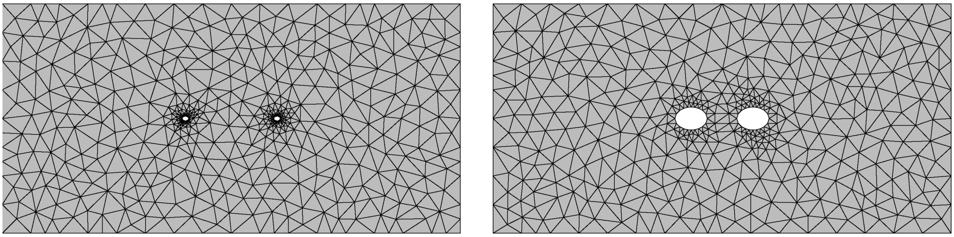
(1) For any vertex x of that mesh, compute $X = \varphi(\frac{x-x_\varepsilon^\pm}{\varepsilon})$.
(2) Find the element K of the bounded mesh used for the profile computation and containing X.
(3) Compute the value $W(X)$ by interpolation in K.

For point (2), a preliminary *bucket sort*, see Ref. 5, pp. 174–177, is performed to reduce the number of elements to be considered when finding K.

To compare u_ε and its zeroth and first order approximations, we need to compute u_ε accurately. Figure 8 shows two meshes used to that end, they have been generated using Triangle Ref. 16. In Fig. 9, we present the differences $u_\varepsilon - u_0$ and $u_\varepsilon - u_1$ on the example of two ellipses. The value $\varepsilon = 0.0585$ is relatively large for visibility reasons, but nevertheless the approximation given by the first order approximation u_1 is much better than u_0. The principal error in $u_\varepsilon - u_0$ is mainly concentrated around the holes, it is partially corrected in $u_\varepsilon - u_1$. We emphasize the fact that, for such values of ε and α ($\alpha = 0.5$), the distance between the two inclusions is $2\varepsilon^\alpha \simeq 0.24$ which is pretty coarse. In this situation, it would be preferable to write the following first order approximation instead of (5.2)

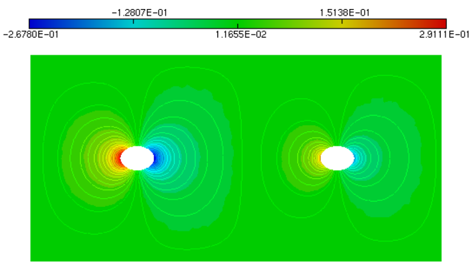
$$u_{11}(x) = u_0(x) + \varepsilon\left[\nabla u_0(x_\varepsilon^-)\cdot \mathbf{V}_{\omega^-}\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + \nabla u_0(x_\varepsilon^+)\cdot \mathbf{V}_{\omega^+}\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right]. \quad (5.6)$$

It appears clearly in Fig. 9 that u_{11} is a better choice than u_1 since the profiles are more precisely corrected near the inclusions.

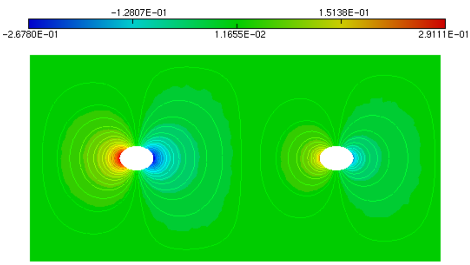


(a) (b)

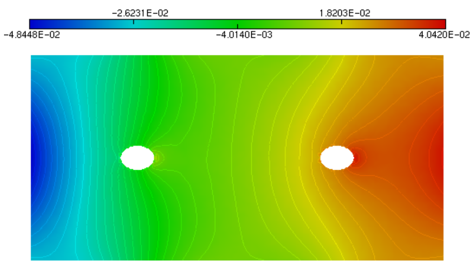
Fig. 8. Some meshes used to compute u_ε. (a) $\alpha = 0.5$ and $\varepsilon = 0.01$, (b) $\alpha = 0.9$ and $\varepsilon = 0.05$.



(a) $u_\varepsilon - u_0$.



(b) $u_\varepsilon - u_1$.



(c) $u_\varepsilon - u_{11}$.

Fig. 9. $u_\varepsilon - u_0$ and $u_\varepsilon - u_1$ for $\varepsilon = 0.0585$ and $\alpha = 0.5$.

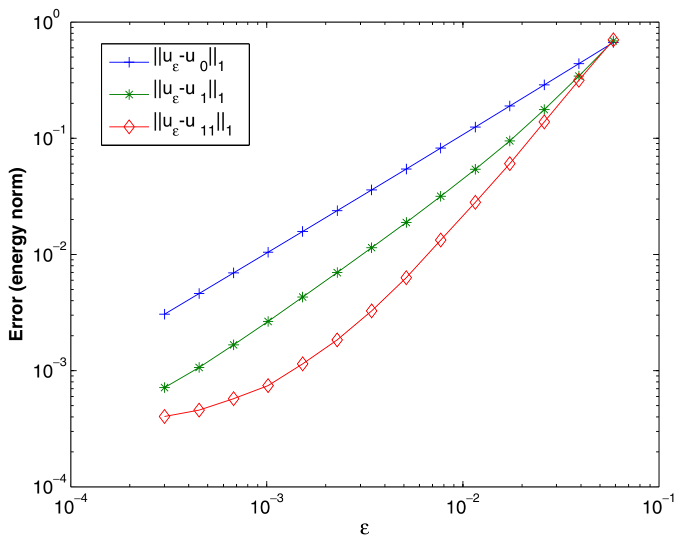


Fig. 10. Energy norms of $u_\varepsilon - u_0$, $u_\varepsilon - u_1$ and $u_\varepsilon - u_{11}$ for $\alpha = 0.2$.

In Fig. 10, we present the errors (in the $H^1(\Omega_\varepsilon)$-norm) obtained for the three approximations u_0, u_1 and u_{11} (in the case where the two inclusions are ellipses, and $\alpha = 0.2$). The local convergence rates computed as the slopes between two consecutive points in Fig. 10 are gathered in Table 1. We recover the expected rate

Table 1. Local convergence rates for curves in Fig. 10.

$u_\varepsilon - u_0$	$u_\varepsilon - u_1$	$u_\varepsilon - u_{11}$
1.0348	1.7149	1.9760
1.0357	1.6385	2.0357
1.0333	1.5339	2.0395
1.0300	1.3843	1.8937
1.0265	1.3183	1.8417
1.0231	1.2786	1.8420
1.0198	1.2351	1.6210
1.0175	1.2145	1.4146
1.0152	1.1975	1.1711
1.0129	1.1897	1.0625
1.0113	1.1481	0.6392
1.0091	1.1072	0.5619
1.0086	0.9804	0.3101

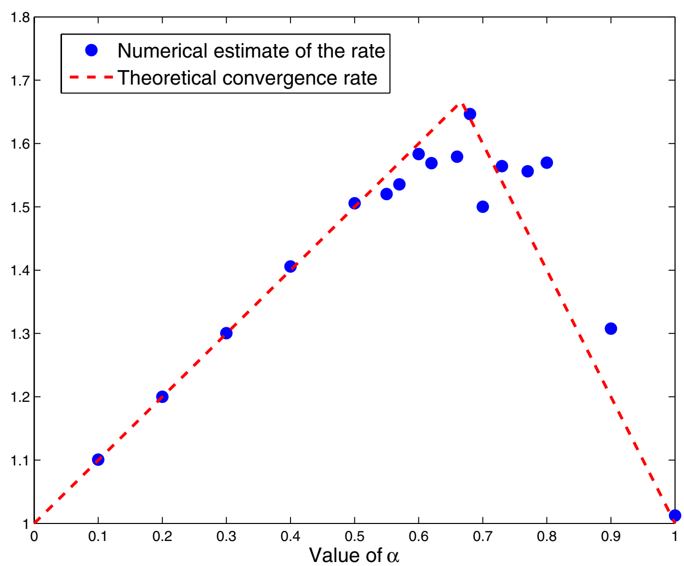


Fig. 11. Predicted and estimated convergence rate with respect to α.

$1 + \alpha = 1.2$ for $u_\varepsilon - u_1$, cf. expansion (2.1), as well as the rate 2 for $u_\varepsilon - u_{11}$ if ε is not too small, cf. expansion (1.5).

Finally, Fig. 11 plots the estimated rates with respect to the value of α. The results are in good agreement with the theoretical predictions. Note that this graph has been obtained for circular holes, where the profile is analytically known, to avoid roundoff errors due to the profile computations.

Acknowledgment

This work has been supported by the ANR (Agence Nationale de la Recherche), project MACADAM number JCJC06-139561.

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